Smooth visibility from a point

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Abstract- We propose a solution in dual space to the problem of computing the tangents of a plane Bezier curve that pass through a point, an important problem in the analysis of visibility. Our solution reduces to the intersection of a line with a Bezier curve in dual space. It is equivalent in complexity to the standard solution, but introduces an alternative, geometric interpretation of the problem. This new characterization has promise for the solution of more general problems in visibility. We also compare the dual solution to the solution using polar theory, which reduces to the intersection of two algebraic curves after implicitization.

Keywords: visibility, tangents through a point, duality, polar.

1 Introduction

The tangents of a plane curve that meet a point P (Figure 1) define important boundaries of visibility from P for robotics, graphics, and geometric analysis. A shortest path from P will begin along one of these tangents (Figure 2). A point light at P will cast shadows bounded by these tangents, when the curve is interpreted as an opaque obstacle (Figure 3). The 2-dimensional silhouette from the viewpoint P is defined by the visible points of tangency (Figure 1).

The solution to the discrete polygonal version of this problem, where the curve is replaced by a polygon, is straightforward (Figure 4). For each vertex of the polygon, consider the outward-pointing normals of its two adjacent edges. Vertices such that one outward-pointing
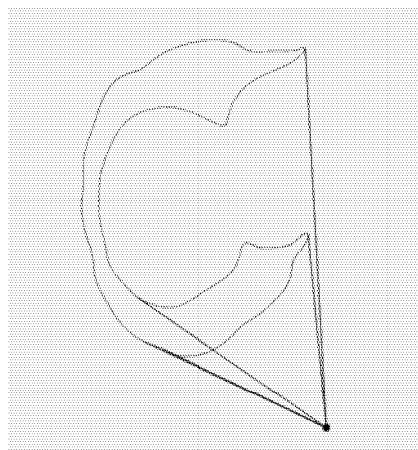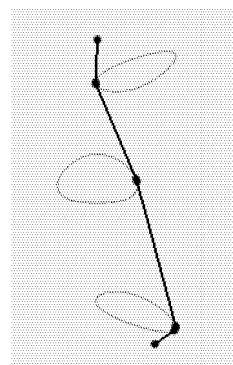

Figure 1: Tangents through a point
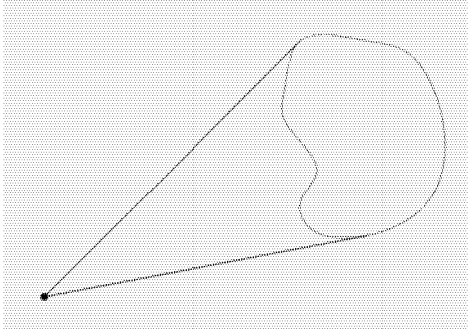

Figure 2: A shortest path

Figure 3: Casting a shadow

Figure 4: The polygonal problem

Figure 5: As normals orthogonal to the line to P
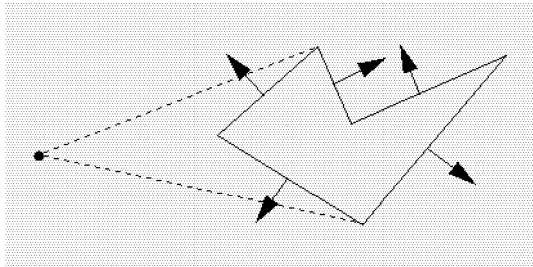
normal points towards the point P and the other points away become the discrete analog of the points of tangency on the curve. A smooth solution using curves is more involved, yet preferable for its precision and, if properly implemented, its improved efficiency over a densely sampled polygon. We are especially interested in the solution of our problem for Bezier curves, since they are the dominant modeling tool for curves.

The classical solution to the tangents of a curve C through a point P uses polar theory [10]. The points of the curve whose tangents intersect P are the intersections of two algebraic curves: the curve C and the polar of P with respect to C (Figure 7).

The standard parametric solution (e.g., [4]) solves for the points $C(t) = (x(t), y(t))$ of the curve whose normal $N(t)$ is orthogonal to the line to $P = (p_1, p_2)$ (Figure 5):

$$(C(t) - P) \cdot N(t) = 0 \qquad (1)$$

or

$$f(t) = (x(t) - p_1, y(t) - p_2) \cdot (-y'(t), x'(t)) = 0$$

This is a univariate equation of degree $2n - 1$ in t, where n is the degree of $C(t)$.

In this paper, we propose a solution that reduces to the intersection of a line and a Bezier curve of degree $2n - 1$ in dual space. This compares favourably with the polar solution. A curve-curve intersection is replaced by a simpler line-curve intersection. Also, the
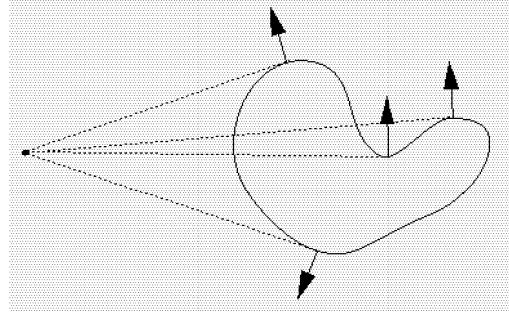
dual of a Bezier curve can be easily represented as a rational Bezier curve (Theorem 3.4), while the polar of a curve cannot. Consequently, the intersection of implicit algebraic curves in the polar solution is replaced by the more familiar intersection of parametric Bezier curves.

The dual solution is similar in efficiency to the standard parametric solution (1), since the intersection of a line and a parametric curve of degree $2n - 1$ is comparable to the solution of a univariate equation of degree $2n-1$. The major difference is in point of view and generalizability. The dual method introduces a new paradigm with great potential for the solution of visibility problems in dual space, in which an algebraic solution is replaced by a geometric solution. In other work, we will show that it can lead to a better solution for the common tangents of two curves [3] (and we shall appeal to some of the results from this paper). We are also presently working on a solution to the common tangent planes of two surfaces that has no comparable solution under the algebraic interpretation. By studying the dual solution for the simpler problem of the tangents through a point, we establish a template for the dual space method and clearly distinguish the geometric and algebraic approaches.

The basic argument of our solution in dual space is as follows. In dual space, points become lines and lines become points. In particular, the point P dualizes to the line P^* and the tangent lines of a curve dualize to the points of a so-called tangential curve (Figure 6). In dual space, the intersections of the line P^* with the tangential curve encode the tangent lines through P (Figures 8-10). In particular, an intersection point maps back to a line, which must be a tangent (since the intersection lies on the tangential curve) that contains P (since the intersection also lies on P^*).

We review the polar solution in the next section, and present our algorithm, including several examples, in Section 3. The paper ends with some conclusions and suggestions for future work. Throughout this paper, we shall work in projective 2-space. We recall some of the important properties of projective 2-space in the follow-
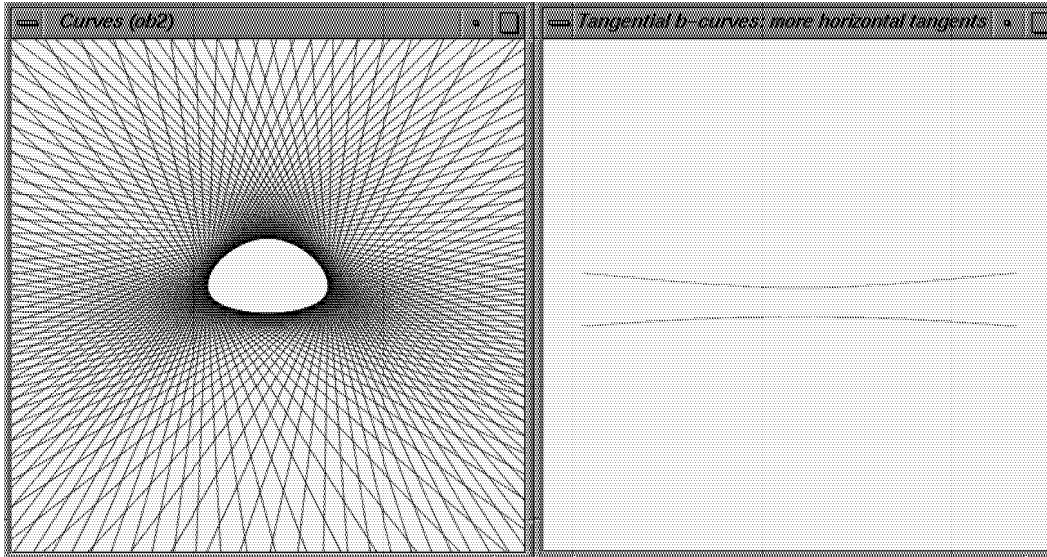
Figure 6: The tangent space of a curve, and its dual (clipped to $x \in [-1, 1]$)

ing definition.

Definition 1.1 Projective 2-space P^2 *is the space* $\{(x_1, x_2, x_3) : x_i \in \Re,$ *not all zero*$\}$ *under the equivalence relation* $(x_1, x_2, x_3) = k(x_1, x_2, x_3)$, $k \neq 0 \in \Re$. *The point* (x_1, x_2, x_3) *in projective 2-space is equivalent to the point* $(\frac{x_1}{x_3}, \frac{x_2}{x_3})$ *in Cartesian 2-space. Points of* P^2 *with* $x_3 = 0$ *are associated with points at infinity. The 3 coordinates in projective 2-space are called* **homogeneous coordinates**. *We distinguish the third homogeneous coordinate by calling it the* **projective coordinate**.

The use of projective space is motivated by the fact that the equation of a line is invariant under multiplication by a constant: the line $ax + by + c = 0$ is equivalent to the line $kax + kby + kc = 0$ for $k \neq 0$.

2 Polar theory

The classical mathematical method for finding the tangents of a curve C through a point P involves polar theory.

Definition 2.1 *Let* C *be a plane algebraic curve defined by the polynomial* $f(x_1, x_2, x_3) = 0$ *of degree* n *and let* $P = (p_1, p_2, p_3)$ *be a point, where both* C *and* P *are expressed in projective 2-space. The* **(first) polar of** P **with respect to** C *is an algebraic curve of degree* $n - 1$ *defined by*

$$p_1 \frac{\partial f}{\partial x_1} + p_2 \frac{\partial f}{\partial x_2} + p_3 \frac{\partial f}{\partial x_3} = 0$$

or $P \cdot \nabla f = 0$ *[10]*.

Example 2.2 *The first polar of* $P = (5, 0, 1)$ *with respect to the circle* $x_1^2 + x_2^2 - x_3^2 = 0$ *is the line* $5(2x_1) - 2x_3 = 0$ *or* $x_1 = \frac{1}{5}$ *in Cartesian space (Figure 7). In this special case of a circle, the first polar can also be computed using inversion: it is the line through the inverse* P' *of* P *and perpendicular to* PP'.

The first polar of $(1, 1, 1)$ *with respect to the quartic trisectrix curve* $(x^2 + y^2 - 2ax)^2 = a^2(x^2 + y^2)$ *[5] is the cubic curve* $p(x, y) = (4 - 4a)(x^3 + xy^2) + 4(y^3 + x^2y) + (6a^2 - 12a)x^2 - (4a + 2a^2)y^2 - 8axy + 6a^2x - 2a^2y = 0$.

Theorem 2.3 *The intersections of* C *and the first polar of* P *with respect to* C *define the locus of points of* C *whose tangents intersect* P *(Figure 7).*
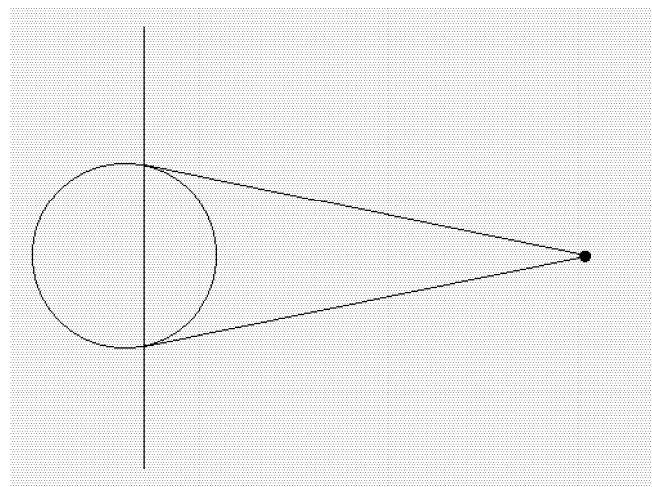

Figure 7: The polar of a point with respect to a circle

Polar theory leads to the following algorithm for the computation of the tangents from a Bezier curve C through a point P:

1. Implicitize C (i.e., translate from the parametric to the implicit representation of C), yielding $f(x_1, x_2, x_3) = 0$.

2. Compute the first polar of P with respect to C, say $g(x_1, x_2, x_3) = 0$.

3. Intersect the algebraic curves $f(x_1, x_2, x_3) = 0$ and $g(x_1, x_2, x_3) = 0$.

The first step, implicitization, involves resultants [6, 9]. The implicitization of a Bezier spline is particularly challenging. For example, the Bezier spline of Figure 1 has 69 segments, yielding a set of algebraic curves.

For steps (2) and (3), Sederberg [8] has developed a robust and efficient algorithm for the computation of polars of piecewise algebraic curves, algebraic curves in the Bernstein basis, as well as the subsequent intersection of the algebraic curve and its polar.

In conclusion, the polar method is awkward for parametric Bezier curves, since it is fundamentally a method for implicit curves.

3   A dual solution

In this section, we present our algorithm in dual space for computing the tangents through a point. This algorithm works directly with Bezier curves. Our basic algorithm for computing the tangents from a curve C through a point P is as follows:

(1) Dualize P and the tangent space of C, to the line P^* and the tangential curve C^*.

(2) Intersect the line P^* and curve C^* in dual space.

(3) Map the intersections in dual space back to lines in primal space. These are the tangents of C through P.

We shall now elaborate on the components of this algorithm and add refinements to handle points at infinity.

We begin with a definition of our duality. Although the most natural point-line duality is between the line $ax + by + c = 0$ and the point (a, b, c) in projective space [2], we choose two subtly different dualities that map a different coefficient of the line equation to the projective coordinate. This choice is motivated by the fact that lines dualized to points with zero projective coordinates will be mapped to infinity (Definition 1.1) and effectively lost by the duality.

Definition 3.1 *In the a-duality, the line $ax + by + c = 0$ is dual to the point $(b, c, a) \in P^2$. In the b-duality, the line $ax + by + c = 0$ is dual to the point $(a, c, b) \in P^2$. The* **tangential a-curve** *(resp., b-curve) of a plane curve $C(t)$ is the curve $C^*(t) \subset P^2$ where $C^*(t)$ is the a-dual (resp., b-dual) of the tangent at $C(t)$. Properties of the tangential curve are discussed in [3].*

The a-dual of the horizontal line $by + c = 0$ is $(b, c, 0)$. That is, horizontal lines are mapped to infinity and the tangential a-curve is not a robust representation of horizontal tangents. Consequently, we restrict the tangential a-curve to the representation of steep tangents, as follows.

Definition 3.2 *A curve's tangent space can be divided into shallow and steep tangents. A tangent $ax + by + c = 0$ is* **steep** *if $|a| \geq |b|$ (its angle to the x-axis is greater than 45 degrees). A tangent is* **shallow** *if $|a| < |b|$.*

Lemma 3.3 *The tangential a-curve clipped by $x = \pm 1$, $C^* \cap \{(x, y) : x \in [-1, 1]\}$, represents the steep tangents of the curve C.*

Proof: Diagonal lines $ax \pm ay + c = 0$ form the boundary between steep and shallow lines. They dualize to $(\pm a, c, a) \in P^2$, or $x = \pm 1$ in Cartesian space, for both a-duality and b-duality. ■

The top right of Figures 8-10 illustrates clipped tangential a-curves (of the curves on the left of each figure). It is simple to clip a curve by $x \in [-1, 1]$.

The tangential b-curve is a perfect complement to the a-curve. Although b-duality maps vertical lines to infinity, the tangential b-curve clipped by $x = \pm 1$ is a robust representation of the shallow tangent space. Thus, we compute the steep tangents through P using the a-curves and the shallow tangents through P using the b-curve, as follows.

(1a) Dualize P and the tangent space of C, using a-duality, to P_a^* and C_a^*. Clip C_a^* to $x \in [-1, 1]$.

(1b) Dualize P and the tangent space of C, using b-duality, to P_b^* and C_b^*. Clip C_b^* to $x \in [-1, 1]$.

(2a) Intersect the line P_a^* and the clipped curve C_a^* in a-dual space.

(2b) Intersect the line P_b^* and the clipped curve C_b^* in b-dual space.

(3a) Dualize the intersections in a-dual space back to steep tangents through P in primal space.

(3b) Dualize the intersections in b-dual space back to shallow tangents through P in primal space.

In summary, the tangents through P are calculated by two line-curve intersections, one in a-space and one in b-space. The two intersections mesh perfectly to generate all of the tangents through P, in a mutually exclusive manner. The clipping of the curves to $x \in [-1, 1]$ makes the intersection very efficient, since intersection with a Bezier curve involves subdivision, which is related to the length of the curve.

Figures 8-10 offer examples of the dual algorithm, varying from simple to complicated. In each case, primal space with the tangents of C through P is illustrated on the left, a-dual space with the tangential a-curves and dual line P_a^* is illustrated on the top right, and b-dual space with the tangential b-curves and dual line P_b^* is illustrated on the bottom right.

The individual steps of the algorithm, dualization and intersection, require some elaboration. A crucial fact is that the tangential curve of a Bezier curve can be expressed as a rational Bezier curve. This simplifies the intersection in dual space, since intersection of Bezier curves is well understood.

Theorem 3.4 *Let $C(t)$ be a plane Bezier curve of degree n with control points $\{(b_{i,1}, b_{i,2})\}_{i=0}^{n}$ over the parameter interval $[t_1, t_2]$. The tangential a-curve $C_a^*(t)$ is a rational Bezier curve of degree $2n - 1$ over $[t_1, t_2]$ with weights $\{w_k\}_{k=0}^{2n-1}$ where:*

$$w_k = \alpha \sum_{j=max(0,k-n)}^{min(n-1,k)} \binom{n-1}{j}\binom{n}{k-j}(b_{j,2} - b_{j+1,2})$$

and control points $\{c_k\}_{k=0}^{2n-1}$ where:

$$c_k = \frac{\alpha}{w_k} \left(\frac{\sum_{j=max(0,k-n)}^{min(n-1,k)} \binom{n-1}{j}\binom{n}{k-j}(b_{j+1,1} - b_{j,1})}{\sum_{\substack{0 \le i \le n-1 \\ 0 \le j \le n \\ i+j=k}} \binom{n-1}{i}\binom{n}{j}\beta_{i,j}} \right)$$

where $\alpha = \frac{n}{(t_2-t_1)\binom{2n-1}{k}}$, and $\beta_{i,j} = (b_{i+1,2}-b_{i,2})b_{j,1} - (b_{i+1,1} - b_{i,1})b_{j,2}$.

Proof: We omit the proof due to lack of space and its technical nature. The reader is referred to [3] for a full proof. ∎

The representation of the tangential b-curve as a rational Bezier curve is analogous.

The line-curve intersection of steps (2a) and (2b) is the intersection of two Bezier curves. It is simple to represent a line as a Bezier curve of degree 1, and intersection with a line is a particularly simple case of Bezier curve intersection. The intersection of Bezier curves is

well understood and efficient [7]. We use the classical subdivision approach, where the two curves are recursively subdivided into shorter segments until their intersection can be adequately approximated by the intersection of two lines.

We end this section by explaining why we did not use the duality $ax + by + c = 0 \rightarrow (a, b, c)$. It maps lines through the origin to infinity. Since a horizontal line can pass through the origin, as can a vertical line, this duality does not cooperate well with either of the other dualities.

4 Conclusions

In this paper, we have studied a new approach to computing the tangents through a point, a basic visibility operation. An algebraic method that reduces to the solution of a univariate equation of degree $2n - 1$ was replaced by a geometric method that reduces to the intersection of a line and Bezier curve of degree $2n - 1$ in dual space. Although, in the context of this problem, the two approaches are similar, the dual method has the better capacity for generalization and other important visibility problems, especially for motion and lighting in 3-space, can be attacked anew using the dual approach. By studying the dual solution for the tangents through a point, we see the development of this new method as it mutates away from the algebraic solution.

The simplicity of the dual method of this paper makes it feasible for many visibility computations from a point to be lifted from an approximate polygonal world to a more realistic curved world. The next step is the application of this smooth tangent operation to the construction of smooth visibility graphs, shortest path motion amongst smooth obstacles, lighting of smooth environments, and other visibility analysis in a smooth curved world.

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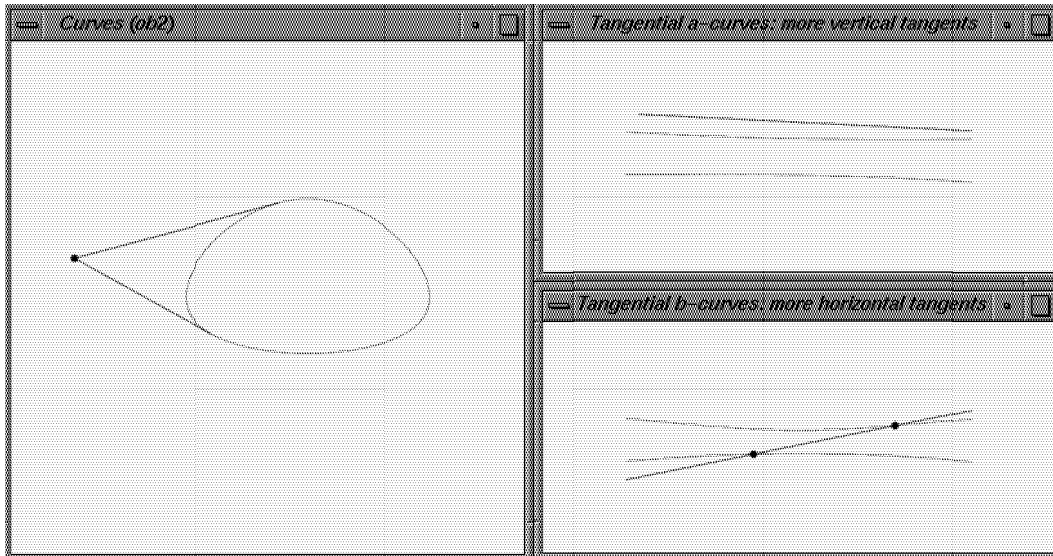
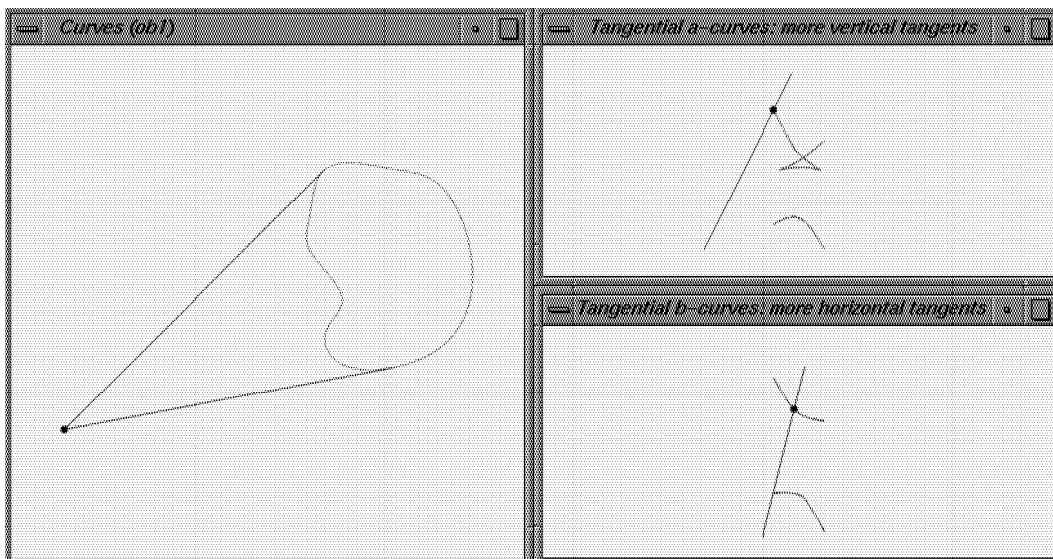Figure 8: A simple example of our algorithm
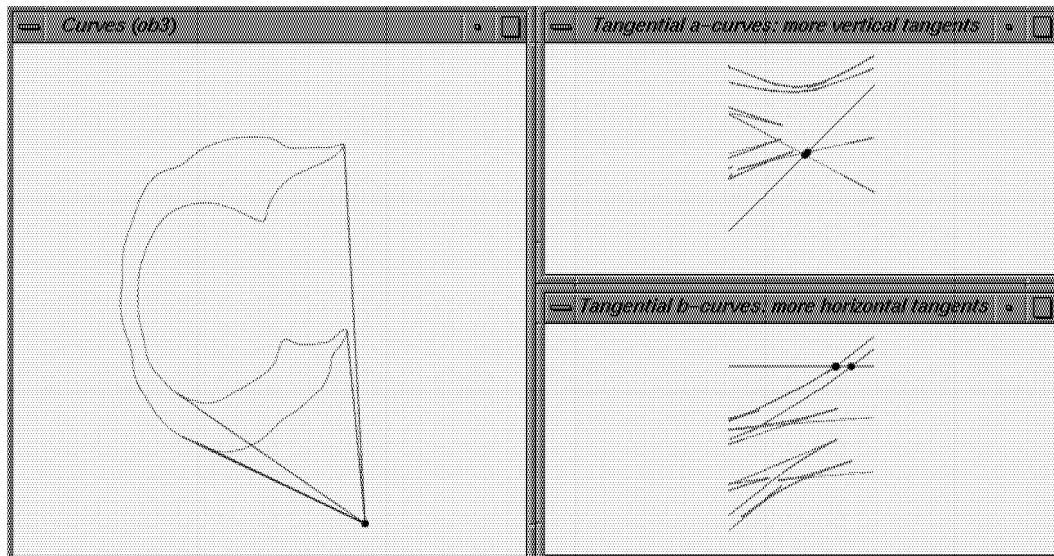

Figure 9: A second example

Figure 10: A more complicated example

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