Computing point/curve and curve/curve bisectors

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1 Introduction

The *bisector* of two geometrical elements (points, curves, surfaces, etc.) is the locus described by a variable point that moves so as to remain equidistant with respect to those elements. The parabola is perhaps the simplest non-trivial example of a bisector — recall [4] its descriptive definition as the locus of a point that maintains equal distances from a given point (the *focus*) and a given straight line (the *directrix*). Although bisectors arise naturally in a variety of algorithms concerned with the decomposition and shape analysis of areas and volumes [2, 15, 19, 24], there has been no systematic investigation of their algebraic and geometric character in the CAD literature.

In this paper we review recent work [8] on the formulation of point/curve bisectors, and present some preliminary results on the more difficult problem of curve/curve bisectors that makes extensive use of the point/curve bisector machinery. Specifically, we consider curve/curve bisectors as the *envelopes* of one-parameter families of point/curve bisectors defined by simultaneously considering one curve in its entirety and a discrete point that moves along the other. This approach is motivated by an attractive property of point/curve bisectors — namely, they can be represented as "variable-distance offsets" to the given curve that (unlike fixed-distance offsets [9, 10]) are *generically* rational curves if the given curve is polynomial or rational. They are thus compatible with the canonical representations of most CAD systems.

Before proceeding, we must clarify what is meant by the *distance* between a point and a curve:

Definition 1.1 *The distance of a point* $\mathbf{q}$ *from a regular parametric curve* $\mathbf{r}(u)$ *defined on the parameter interval* I *is given by*

$$\mathrm{dist}(\mathbf{q},\mathbf{r}(u)) \;=\; \inf_{u\in I}\,|\,\mathbf{q}-\mathbf{r}(u)\,|\,. \tag{1.1}$$

Although we confine our attention here to parametric curves, most of the concepts extend straightforwardly to implicitly-defined curves also. A curve $\mathbf{r}(u)$ is said to have a *regular* parameterization if its derivative $\mathbf{r}'(u) = \mathrm{d}\mathbf{r}/\mathrm{d}u$ is non-vanishing over the domain of definition, $u \in I$ (this guarantees that the curve locus is smooth [7], although it may exhibit self-intersections).

If $\mathbf{r}(u)$ is a polynomial curve, the bound (1.1) will always be attained at a *finite* parameter value u regardless of whether I is finite or infinite. If $\mathbf{r}(u)$ is a rational curve and I is not finite, however, it is possible that (1.1) will be attained only in the limit $|u| \to \infty$.

Remark 1.1 For the point $\mathbf{q} = (a, b)$ and the regular polynomial curve $\mathbf{r}(u) = \{X(u), Y(u)\}$ of degree n defined on $u \in I$, let $u_1, \ldots, u_N$ be the distinct *odd*-multiplicity roots of the polynomial

$$P_\perp(u) = [\,a - X(u)\,]\,X'(u) + [\,b - Y(u)\,]\,Y'(u) \tag{1.2}$$

of degree $2n-1$ on the interior of the interval I, augmented by the finite end points (if any) of I. Then the distance function (1.1) may be expressed as

$$\mathrm{dist}(\mathbf{q}, \mathbf{r}(u)) = \min_{1 \leq k \leq N} |\,\mathbf{q} - \mathbf{r}(u_k)\,|\,. \tag{1.3}$$

An analogous result holds for regular rational curves, provided we replace the odd roots of the polynomial (1.2) by those of

$$\begin{aligned} P_\perp(u) = {} & [\,aW(u) - X(u)\,]\,[\,W(u)X'(u) - W'(u)X(u)\,] \\ & + [\,bW(u) - Y(u)\,]\,[\,W(u)Y'(u) - W'(u)Y(u)\,] \end{aligned} \tag{1.4}$$

satisfying $W(u) \neq 0$ on the interval I. Geometrically, roots of $P_\perp(u)$ identify points of the curve where lines drawn from $\mathbf{q}$ meet $\mathbf{r}(u)$ orthogonally. The distance (1.1) is simply the smallest of the lengths of these perpendiculars (and the chords drawn from $\mathbf{q}$ to the affine end points of $\mathbf{r}(u)$, if any). Even-multiplicity roots of $P_\perp(u)$ are ignored since they identify points of $\mathbf{r}(u)$ where the distance $|\,\mathbf{q} - \mathbf{r}(u)\,|$ "levels off" but then continues to increase or decrease (*i.e.*, it does not attain a local extremum).

Proposition 1.1 For a regular curve $\mathbf{r}(u)$ the function $f(\mathbf{q}) = \mathrm{dist}(\mathbf{q}, \mathbf{r}(u))$ is everywhere continuous, but not always differentiable, with respect to $\mathbf{q}$.

Proof : See [8, Proposition 1.2]; also [17]. ∎

2 Point/curve bisectors

Recall [9, 10] that the "untrimmed" offset at fixed distance d to a regular plane parametric curve $\mathbf{r}(u)$ with unit normal $\mathbf{n}(u)$ is the locus defined by

$$\mathbf{o}(u) \,=\, \mathbf{r}(u) + d\,\mathbf{n}(u)\,. \tag{2.1}$$

The *true* offset to $\mathbf{r}(u)$ at distance d is obtained from (2.1) by "trimming" away certain parametric subsegments of the latter [9, 10]. (While, for each ξ, corresponding points $\mathbf{o}(\xi)$ and $\mathbf{r}(\xi)$ are distance d apart measured along their mutual normal $\mathbf{n}(\xi)$, the point $\mathbf{o}(\xi)$ of the untrimmed offset is not necessarily distance d from the *entire curve* $\mathbf{r}(u)$ — see [9, 10] for further details.)

We define the "untrimmed" bisector of a fixed point $\mathbf{p}$ and a regular plane curve $\mathbf{r}(u)$ as a generalization of (2.1), replacing the fixed offset distance d by a *displacement function*:

$$\mathbf{b}(u) \,=\, \mathbf{r}(u) + d(u)\mathbf{n}(u)\,. \tag{2.2}$$

For $\mathbf{b}(u)$ to represent the untrimmed bisector of $\mathbf{p}$ and $\mathbf{r}(u)$, the appropriate choice is

$$d(u) \,=\, \frac{|\,\mathbf{p}-\mathbf{r}(u)\,|^2}{2\,(\mathbf{p}-\mathbf{r}(u))\cdot\mathbf{n}(u)}\,. \tag{2.3}$$

The reader can easily verify that, for each u, the displacement (2.3) identifies the *unique* point along the curve normal line at $\mathbf{r}(u)$ that is equidistant from the given point $\mathbf{p}$ and the curve point $\mathbf{r}(u)$. Note also that $d(u) \neq 0$ for all u when $\mathbf{p}$ does not lie on $\mathbf{r}(u)$. Figure 1 illustrates the formulation of untrimmed point/curve bisectors as "variable–distance" offsets.

Remark 2.1 From expressions (2.2) and (2.3) we see that when the given curve $\mathbf{r}(u) = \{x(u), y(u)\}$ has a polynomial or rational parameterization, the untrimmed bisector $\mathbf{b}(u)$ is a *rational* curve — since there is an obvious cancellation of the radical $\sqrt{x'^2(u)+y'^2(u)}$ incurred in the unit normal $\mathbf{n}(u)$.

We now briefly enumerate some key properties of untrimmed point/curve bisectors (full details may be found in [8]; see also [13]).

2.1 Parameterization of the untrimmed bisector

Let $X(u), Y(u), W(u)$ be polynomials of degree n, and let $\mathbf{b}(u)$ denote the untrimmed bisector of the point $\mathbf{p} = (\alpha, \beta)$ and the regular polynomial curve $\mathbf{r}(u) = \{X(u), Y(u)\}$ or rational curve $\mathbf{r}(u) = \{X(u)/W(u), Y(u)/W(u)\}$.

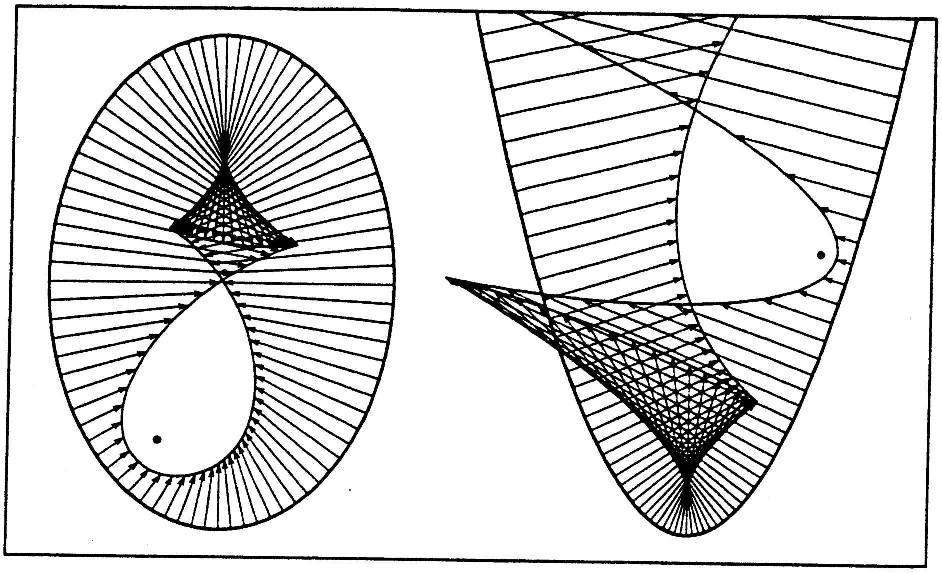


FIGURE 1. Examples of untrimmed point/curve bisectors regarded as "variable–distance" offset curves.

Writing $\mathbf{b}(u) = \{X_b(u)/W_b(u), Y_b(u)/W_b(u)\}$, from (2.2) and (2.3) we can express the homogeneous coordinates of the untrimmed bisector as

$$\begin{aligned} X_b &= [\,\alpha^2 - X^2 + (\beta - Y)^2\,]\,Y' \,-\, 2(\beta - Y)XX' \,, \\ Y_b &= 2(\alpha - X)YY' \,-\, [\,(\alpha - X)^2 + \beta^2 - Y^2\,]\,X' \,, \\ W_b &= 2\,[\,(\alpha - X)Y' - (\beta - Y)X'\,] \end{aligned} \tag{2.4}$$

in the case of a polynomial curve, and

$$\begin{aligned} X_b &= [\,\alpha^2 W^2 - X^2 + (\beta W - Y)^2\,]\,V \,-\, 2(\beta W - Y)XU \,, \\ Y_b &= 2(\alpha W - X)YV \,-\, [\,(\alpha W - X)^2 + \beta^2 W^2 - Y^2\,]\,U \,, \\ W_b &= 2W\,[\,(\alpha W - X)V - (\beta W - Y)U\,] \end{aligned} \tag{2.5}$$

(where we set $U = WX' - W'X$ and $V = WY' - W'Y$) in the rational case.

Remark 2.2 From the above it may be verified that $\mathbf{b}(u)$ is of degree $3n-1$ or $4n-2$, at most, when $\mathbf{r}(u)$ is a polynomial or rational curve of degree n.

Expressions (2.4) and (2.5) can, of course, be formulated to yield control points for $\mathbf{b}(u)$ in the standard Bézier representation over any parameter domain of interest when $\mathbf{r}(u)$ is specified in Bézier form. This is an elementary application of the arithmetic of polynomials in Bernstein form [12] and is desirable in view of the numerical stability [11] of this representation.

2.2 Points at infinity

The untrimmed bisector has (real) points at infinity corresponding to *finite* (real) points of $\mathbf{r}(u)$ when the condition $(\mathbf{p}-\mathbf{r}(u))\cdot\mathbf{n}(u)=0$ is satisfied, *i.e.*, the line joining $\mathbf{p}$ to $\mathbf{r}(u)$ is tangential to the curve at that point. Thus, for regular polynomial and rational curves, the roots of the polynomials

$$P_\infty(u) \;=\; [\,\alpha - X(u)\,]\,Y'(u) \;-\; [\,\beta - Y(u)\,]\,X'(u) \tag{2.6}$$

and

$$\begin{aligned} P_\infty(u) \;=\; & [\,\alpha W(u) - X(u)\,]\,[\,W(u)Y'(u) - W'(u)Y(u)\,] \\ & -\; [\,\beta W(u) - Y(u)\,]\,[\,W(u)X'(u) - W'(u)X(u)\,] \end{aligned} \tag{2.7}$$

identify points at infinity on $\mathbf{b}(u)$. In the rational case $\mathbf{b}(u)$ has, additionally, points at infinity corresponding to finite values of u that are roots of $W(u)$.

2.3 Cusps and higher–order irregular points

The untrimmed bisector $\mathbf{b}(u)$ of a point $\mathbf{p}$ and a smooth curve $\mathbf{r}(u)$ is not, in general, a smooth locus itself. Irregular points of the untrimmed bisector are identified by the condition $|\mathbf{b}'(u)|=0$, and by differentiating (2.2) it can be verified [8] that this is satisfied when the curvature $\kappa = |\mathbf{r}'|^{-3}\,(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{z}$ of the given curve $\mathbf{r}(u)$ equals the local "critical" value

$$\kappa_{\rm crit}(u) \;=\; -\,\frac{1}{d(u)}\,. \tag{2.8}$$

Geometrically, equation (2.8) indicates that a point of $\mathbf{b}(u)$ will be irregular if it coincides with the *center of curvature* for the corresponding point of $\mathbf{r}(u)$. Thus, irregular points of the untrimmed bisector must lie on the *evolute* (the locus of centers of curvature) of the curve $\mathbf{r}(u)$, regardless of the location of the point $\mathbf{p}$. Some examples are shown in Figure 2.

It is intriguing to note that while (2.8) appears on casual inspection to be a straightforward generalization of the analogous criterion[1] $\kappa(u)=-1/d$ [9] for irregular points on the constant–distance offset (2.1), this condition is in fact not appropriate to variable–distance offsets with *arbitrary* displacement functions $d(u)$ — it is specific to the particular form given by (2.3), for which $\kappa(u)=-1/d(u) \implies d'(u)=0$.

For polynomial and rational curves, respectively, the parameter values of irregular points on the untrimmed bisector are roots of the polynomials

$$\begin{aligned} P_c \;=\; & [\,(\alpha-X)^2+(\beta-Y)^2\,]\,(X'Y''-X''Y') \\ & +\; 2\,(X'^2+Y'^2)\,[\,(\alpha-X)Y'-(\beta-Y)X'\,] \end{aligned} \tag{2.9}$$

[1] For constant–distance offsets, it can further be shown [9] that their cusps meet the evolute *orthogonally*. From the examples of Figure 2, however, it is evident that cusps of the untrimmed bisector $\mathbf{b}(u)$ are not, in general, orthogonal to the evolute of $\mathbf{r}(u)$.

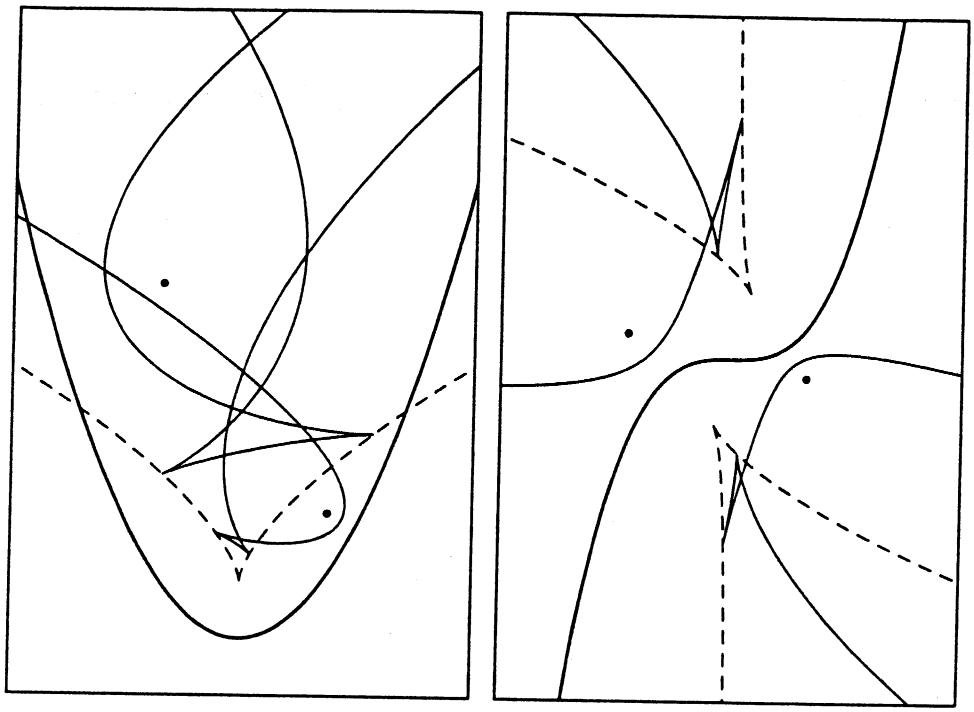


FIGURE 2. Cusps of $\mathbf{b}(u)$ lie on the evolute (dashed curve) of $\mathbf{r}(u)$.

and

$$\begin{aligned} P_c &= W\,[\,(\alpha W - X)^2 + (\beta W - Y)^2\,]\,(U_1V_2 - U_2V_1) \\ &+ 2\,(U_1^2 + V_1^2)\,[\,(\alpha W - X)V_1 - (\beta W - Y)U_1\,] \end{aligned} \tag{2.10}$$

where, for brevity, we have written $(U_1, V_1) = (WX' - W'X, WY' - W'Y)$ and $(U_2, V_2) = (WX'' - W''X, WY'' - W''Y)$ in the latter case.

Remark 2.3 If the curvature of $\mathbf{r}(u)$ attains the critical value (2.8) without being an extremum ($\kappa'(u) \neq 0$), the corresponding irregular point of $\mathbf{b}(u)$ is a *cusp*, *i.e.*, a sudden tangent reversal. However, if the value (2.8) represents a local extremum of $\kappa(u)$ — a "vertex" of the curve $\mathbf{r}(u)$ — then $\mathbf{b}(u)$ will exhibit a *tangent-continuous point of infinite curvature* (see [8]).

2.4 Self–intersections of the untrimmed bisector

It is possible to construct a minimal polynomial $P_i(u)$ whose distinct real roots $u_1, u_2, \ldots$ correspond to self–intersections of the untrimmed bisector, so that $\mathbf{b}(u_j) = \mathbf{b}(u_k)$ for some $j \neq k$. This construction is quite involved, however, and involves a mass of algebraic details that are not germane to our present brief survey. Ultimately, it relies on expanding a modified Sylvester determinant and discarding certain "extraneous" factors from the result — full details may be found in [8].

Remark 2.4 We can deduce[2] the degree of $P_i(u)$ indirectly by noting that the untrimmed bisector $\mathbf{b}(u)$ is a *rational* curve, and invoking the fact [21] that any rational curve of degree m has precisely $\frac{1}{2}(m-1)(m-2)$ double points (or their equivalent). Consider, for example, the case where $\mathbf{r}(u)$ is a polynomial curve of degree n. Then $\mathbf{b}(u)$ is of degree $m = 3n-1$, and thus has $\frac{1}{2}(3n-2)(3n-3)$ double points — including cusps, which occur at the roots of (2.9), a polynomial of degree $4(n-1)$. Thus the number of nodes is $\frac{1}{2}(3n-2)(3n-3)-4(n-1) = \frac{1}{2}(9n^2-23n+14)$. Since two distinct parameters are associated with each node, we infer that $\deg(P_i) = 9n^2-23n+14$. Hence for point/cubic bisectors we have $\deg(P_i) = 26$ in general, although in typical examples one will observe fewer than thirteen real self-intersections.

We assume henceforth that $P_i(u)$, along with the polynomials $P_\infty(u)$ and $P_c(u)$ defined in the preceding discussion, have been determined and that their real roots on the parameter domain of interest have been identified. We emphasize again that the Bernstein–Bézier form is the preferred medium for executing these calculations [6], since even simple curves can give rise to polynomials of relatively high degree. Having thus identified all cusps, points at infinity, and self-intersections of the untrimmed bisector, we can proceed to *trim* $\mathbf{b}(u)$ — *i.e.*, to determine which parametric subsegments (if any) must be discarded to give the "true" point/curve bisector.

2.5 The trimming procedure

From (2.2) and (2.3) we note that the untrimmed bisector of $\mathbf{p}$ and $\mathbf{r}(u)$ can be regarded as the locus of centers of a family of circles that pass through $\mathbf{p}$ and touch — *i.e.*, are tangent to — $\mathbf{r}(u)$ at some point (see Figure 3).

Remark 2.5 Let C_ξ denote the circle that passes through $\mathbf{p}$ and touches $\mathbf{r}(u)$ at $u = \xi$. Then the point $\mathbf{b}(\xi)$ of the untrimmed bisector — the center of the circle C_ξ — belongs to the "true" bisector if and only if no point of the curve $\mathbf{r}(u)$ lies *inside* the circle C_ξ.

The untrimmed bisector $\mathbf{b}(u)$ is "trimmed" down to the true bisector by deleting a finite number of continuous segments. To accomplish this, we must cut $\mathbf{b}(u)$ at certain "special" points that delineate possible deviations of the untrimmed bisector from the true bisector. These special points are the points at infinity, cusps, and self-intersections of $\mathbf{b}(u)$ discussed above.

A point $\mathbf{b}(\xi)$ of the untrimmed bisector may be denied membership in the true bisector under two circumstances: (i) the curve $\mathbf{r}(u)$ crosses the circle C_ξ "locally," *i.e.*, at the point $\mathbf{r}(\xi)$ to which C_ξ is tangent; or (ii) the curve

[2] Note that such a simple deduction cannot be made for the self-intersection polynomial [10] appropriate to the constant-distance offsets (2.1), since these offsets are *not* rational.

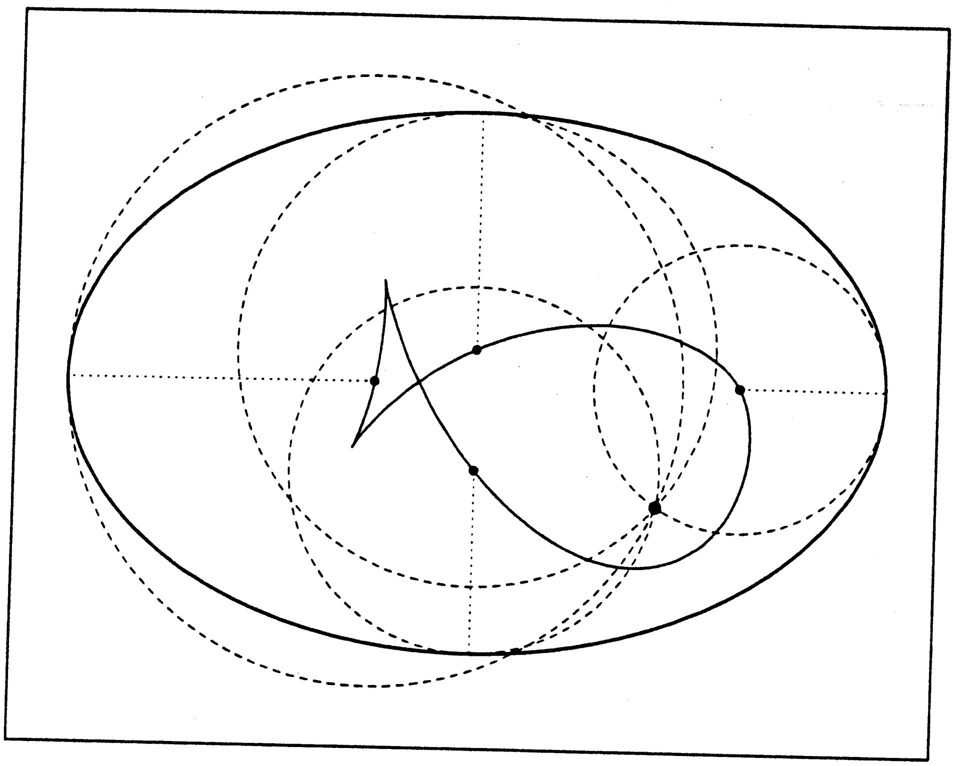


FIGURE 3. Circles that pass through $\mathbf{p}$ and touch $\mathbf{r}(u)$.

$\mathbf{r}(u)$ crosses C_ξ in a "global" sense, *i.e.*, at some other point $\mathbf{r}(\omega)$ totally unrelated to the point of tangency ($\omega \neq \xi$).

We split the trimming process into two stages. In the first stage "inactive" segments of $\mathbf{b}(u)$, which fall under category (i) above, are deleted. The second stage eliminates segments of $\mathbf{b}(u)$ that fall under category (ii).

2.5.1 Active and inactive segments

For a given point $\mathbf{p}$ and curve $\mathbf{r}(u)$, we call $\mathbf{r}(\xi)$ and $\mathbf{b}(\xi) = \mathbf{r}(\xi) + d(\xi)\mathbf{n}(\xi)$ "corresponding" points of the curve and the untrimmed point/curve bisector for each ξ. Note that a given *geometric* point on the untrimmed bisector may have more than one corresponding point on the curve $\mathbf{r}(u)$, *i.e.*, for $\xi_1 \neq \xi_2$ it is possible that $\mathbf{r}(\xi_1) + d(\xi_1)\mathbf{n}(\xi_1) = \mathbf{r}(\xi_2) + d(\xi_2)\mathbf{n}(\xi_2)$.

Now it is clear that certain points $\mathbf{b}(\xi)$ of the untrimmed bisector do not belong to the "true" bisector, because there are points along the curve $\mathbf{r}(u)$ that are closer to these points $\mathbf{b}(\xi)$ than their corresponding points $\mathbf{r}(\xi)$.

Definition 2.1 The point $\mathbf{q} = \mathbf{b}(\xi)$ of the untrimmed bisector is *active* if either of the following conditions holds (see Figure 4):

(1) $\mathbf{q}$ has more than one corresponding point on the curve; or

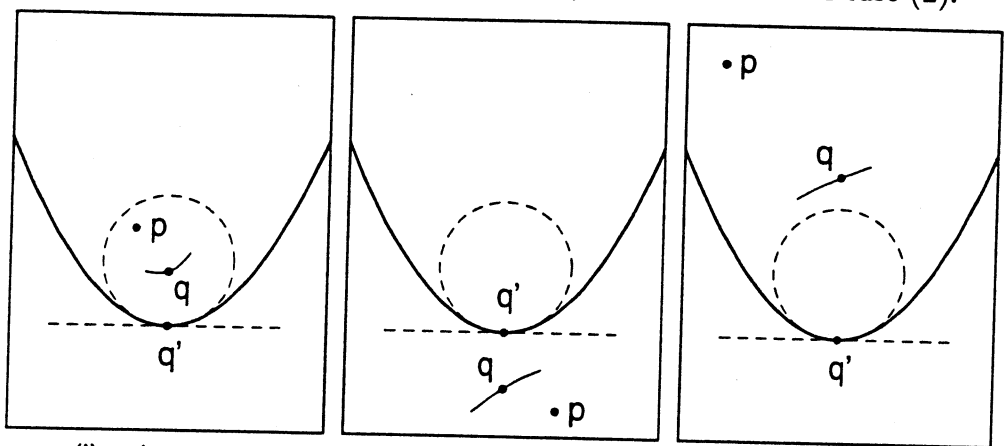
(2) $\mathbf{q}$ has only one corresponding point $\mathbf{q}' = \mathbf{r}(\xi)$ on the curve, and:

(a) the point $\mathbf{p}$ lies on or inside the circle of curvature at $\mathbf{q}'$; or

(b) the point $\mathbf{p}$ and the circle of curvature at $\mathbf{q}'$ lie on opposite sides of the tangent at $\mathbf{q}'$.

A segment S of $\mathbf{b}(u)$ is active if each point of S is active.

Case (1) above corresponds to self–intersections of $\mathbf{b}(u)$, which we shall deal with in the following section. For now, we concentrate on case (2).



(i) q is active (ii) q is active (iii) q is inactive

FIGURE 4. Active and inactive points on the untrimmed bisector.

An active point appears — at least "locally" — to be on the true bisector. In particular, if $\mathbf{b}(\xi)$ is an active point with a unique corresponding point $\mathbf{r}(\xi)$, then in some neighborhood of $u = \xi$ the curve $\mathbf{r}(u)$ lies completely *outside* the circle C_ξ (see Remark 2.5).

To see this, we note that if $\mathbf{p}$ lies inside or on the circle of curvature at $\mathbf{r}(\xi)$, then it can be shown [8, Lemma 3.1] that C_ξ also lies inside or coincides with the circle of curvature at $\mathbf{r}(\xi)$ and, in particular, there exists a neighborhood of $u = \xi$ on $\mathbf{r}(u)$ that lies entirely outside the circle C_ξ [16, p. 176]. On the other hand, if $\mathbf{p}$ (and thus C_ξ) lies on the opposite side of the tangent at $\mathbf{r}(\xi)$ to the circle of curvature there, then again $\mathbf{r}(u)$ in some neighborhood of $u = \xi$ lies entirely outside the circle C_ξ.

Remark 2.6 An inactive point of the untrimmed bisector $\mathbf{b}(u)$ of $\mathbf{p}$ and $\mathbf{r}(u)$ does not belong to their true bisector.

Proof : See [8, Proposition 3.1]. ∎

Since the definition of whether the point $\mathbf{b}(\xi)$ of the untrimmed bisector is active or inactive depends on the location of $\mathbf{p}$ relative to the tangent line and the circle of curvature of $\mathbf{r}(u)$ at $u = \xi$, we are interested in points where these relative locations can change.

Theorem 2.1 *Let* $\mathbf{b}(u)$ *be the untrimmed bisector of a point* $\mathbf{p}$ *and a regular curve* $\mathbf{r}(u)$ *defined on the interval* $u \in I$*, and let* $\{u_1, \ldots, u_M\} \in I$ *be the ordered set of parameter values that correspond to points at infinity or cusps on* $\mathbf{b}(u)$*. Then, denoting the end points of* I *by* u_0 *and* u_{M+1}*, we have either*

$$\mathbf{b}(u) \text{ is active for all } u \in (u_k, u_{k+1}) \tag{2.11}$$

or

$$\mathbf{b}(u) \text{ is inactive for all } u \in (u_k, u_{k+1}) \tag{2.12}$$

on each segment (u_k, u_{k+1}) *for* $k = 0, \ldots, M$ *of* $\mathbf{b}(u)$.

Proof : See [8]. ∎

In the context of the above theorem,[3] we remind the reader that the point $\mathbf{b}(\xi)$ of the untrimmed bisector is (i) a point at infinity if $\mathbf{p}$ lies on the tangent line at $\mathbf{r}(\xi)$; and (ii) a cusp if $\mathbf{p}$ lies on the circle of curvature at $\mathbf{r}(\xi)$.

2.5.2 Trimming at self–intersections

Recall (Remark 2.5) that the point $\mathbf{b}(\xi)$ of the untrimmed bisector will belong to the true bisector if and only if the circle C_ξ centered on $\mathbf{b}(\xi)$ that passes through $\mathbf{p}$ and touches $\mathbf{r}(u)$ at $u = \xi$ has no point of $\mathbf{r}(u)$ in its interior.

Self–intersections $\mathbf{b}(\xi_1) = \mathbf{b}(\xi_2)$ of the untrimmed bisector (where $\xi_1 \neq \xi_2$) correspond to points whose circles C_{ξ_1} and C_{ξ_2} coincide — *i.e.*, there exists a single circle $C = C_{\xi_1} = C_{\xi_2}$ that passes through $\mathbf{p}$ and touches $\mathbf{r}(u)$ at *two* points, $u = \xi_1$ and $u = \xi_2$. As we move along the untrimmed bisector, its self–intersections delineate (possible) transitions from a regime in which the circles C_ξ corresponding to each point $\mathbf{b}(\xi)$ are "empty" to one in which they are "occupied" by points of $\mathbf{r}(u)$, or vice–versa — see [8] for full details.

Thus, the second stage of the trimming process consists of splitting the "active" segments of the untrimmed bisector at its self–intersections, and validating the resulting subsegments for membership in the true bisector (see Theorem 3.2 in [8]). As noted above, computing the parameter values that correspond to self–intersections of $\mathbf{b}(u)$ is not a trivial matter — relatively simple input curves give rise to polynomials $P_i(u)$ of high degree, reflecting the potentially complicated topology (*e.g.*, Figure 5) of the real locus of $\mathbf{b}(u)$.

2.5.3 The algorithm

Given a "robust" polynomial root–solver, we now outline an algorithm, based on the above ideas, for computing the bisector of a point $\mathbf{p}$ and a regular (polynomial or rational) parametric curve $\mathbf{r}(u)$:

[3]Exceptionally, $\mathbf{p}$ may lie simultaneously on both the tangent line and the circle of curvature at $\mathbf{r}(\xi)$ if that point is an *inflection*; but from Definition 2.1 it can be seen that all points of the untrimmed bisector in some neighborhood of $\mathbf{b}(\xi)$ must then be active.

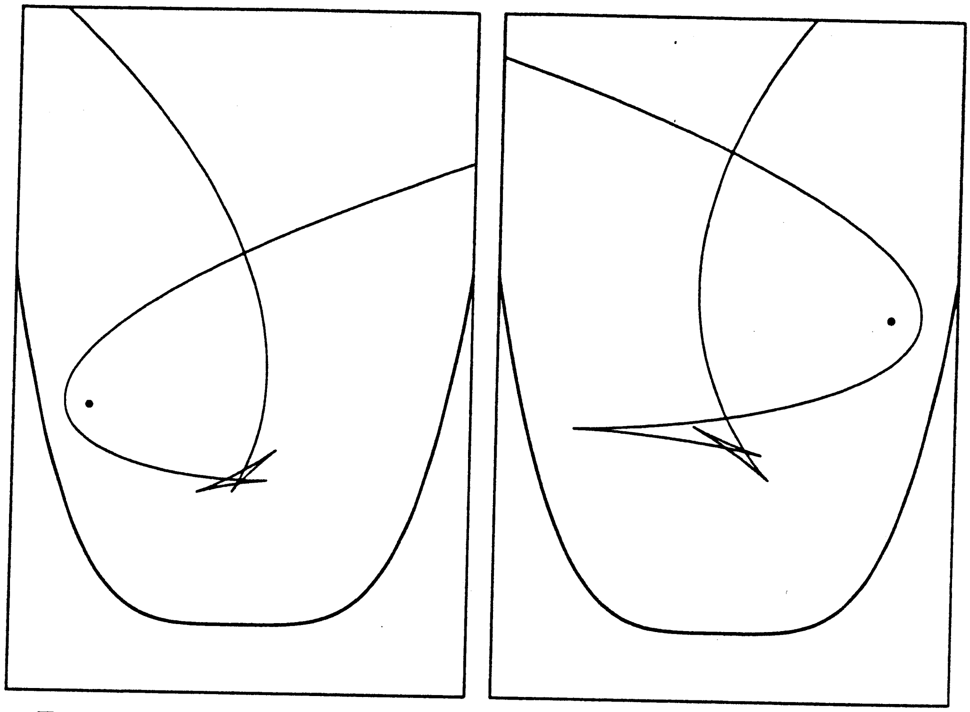


FIGURE 5. Untrimmed bisectors of a point and $\mathbf{r}(u) = \{u, u^4\}$.

1. Formulate the untrimmed bisector $\mathbf{b}(u)$ — as defined by equations (2.4) and (2.5), for polynomial and rational curves, respectively.
2. Compute the points at infinity and cusps of $\mathbf{b}(u)$, *i.e.*, the roots of (2.6) and (2.9) or (2.7) and (2.10), as appropriate.
3. For each segment of $\mathbf{b}(u)$ delineated by these points, compare distances of the segment parametric midpoint to the point $\mathbf{p}$ and to the curve $\mathbf{r}(u)$, using (1.3). Discard segments for which these distances are unequal.
4. Formulate the self-intersection polynomial $P_i(u)$ — as described in [8] — and identify its real roots that lie on the remaining "active" segments.
5. Split each active segment at the self-intersections of $\mathbf{b}(u)$ and compare the distances of the parametric midpoint of each resulting subsegment to $\mathbf{p}$ and to $\mathbf{r}(u)$: discard if these distances are unequal. The remaining rational arcs constitute the true bisector.

Remark 2.7 The true bisector of a point $\mathbf{p}$ and a (finite or infinite) curve $\mathbf{r}(u)$ always encloses a convex, simply-connected region of the plane, which may be of finite or infinite area — see, for example, Figure 7 below. This is because the region in question may be regarded as the set-intersection of a continuous family of half-planes $\mathcal{S}(u)$ where, for each u, $\mathcal{S}(u)$ contains the given point $\mathbf{p}$ and is bounded by the perpendicular bisector of $\mathbf{p}$ and the point $\mathbf{r}(u)$ of the curve. The set–intersection of a family of half–planes always yields a convex set (see [17, Theorem 7 on p. 112]).

In the case of a *finite* curve $\mathbf{r}(u)$, defined on $u \in [\,a, b\,]$ say, it is necessary to "complete" the untrimmed bisector [8] by incorporating the appropriately directed tangent–extensions to $\mathbf{b}(u)$ at its end points $u = a$ and $u = b$. The intersections of these tangent–extensions with $\mathbf{b}(u)$ for $u \in [\,a, b\,]$ and with each other must then be included in the trimming process.

2.6 An illustrative example

To illustrate the above ideas in a concrete setting, we consider the bisector of the point $\mathbf{p} = (\alpha, \beta)$ and the ellipse

$$X(u) \,=\, 1 - u^2\,, \quad Y(u) \,=\, 2k\,u\,, \quad W(u) \,=\, 1 + u^2 \tag{2.13}$$

centered on the origin, with semi–axes 1 and k. From equations (2.5), we see that the untrimmed bisector is a rational curve of degree *six*, defined by

$$\begin{aligned} X_b(u) &= (1-u^2)\,[\,k(\alpha^2+\beta^2-1)u^4 \,+\, 4(1-k^2)\beta u^3 \\ &\quad +\, 2k(\alpha^2+\beta^2-3+2k^2)u^2 \,+\, 4(1-k^2)\beta u \,+\, k(\alpha^2+\beta^2-1)\,]\,, \\ Y_b(u) &= 2u\,[\,(\alpha^2+\beta^2+2(1-k^2)\alpha+1-2k^2)u^4 \\ &\quad +\, 2(\alpha^2+\beta^2-1)u^2 \,+\, \alpha^2+\beta^2-2(1-k^2)\alpha+1-2k^2\,]\,, \\ W_b(u) &= 2(1+u^2)^2\,[\,-k(\alpha+1)u^2 \,+\, 2\beta u \,+\, k(\alpha-1)\,]\,. \end{aligned} \tag{2.14}$$

Examples of the curves defined by equation (2.14) are illustrated in Figure 6, for various locations of the point $\mathbf{p}$.

Up to a constant factor, the polynomial (2.7) in this case reduces to

$$P_\infty(u) \,=\, (u^2+1)\,[\,k(1+\alpha)\,u^2 \,-\, 2\beta\,u \,+\, k(1-\alpha)\,]\,, \tag{2.15}$$

and there are evidently two real points at infinity whenever $\alpha^2 + (\beta/k)^2 > 1$, *i.e.*, $\mathbf{p}$ lies *outside* the ellipse. They correspond to the parameter values

$$u \,=\, \frac{\beta \pm \sqrt{k^2\alpha^2+\beta^2-k^2}}{k(\alpha+1)}\,. \tag{2.16}$$

The cusps of the untrimmed bisector are roots of the polynomial (2.10). Substituting from (2.13), this becomes

$$\begin{aligned} P_c(u) &= k\,[\,\alpha^2+\beta^2+2(1-k^2)\alpha+1-2k^2\,]\,u^6 \\ &\quad +\, 3k\,[\,\alpha^2+\beta^2-2(1-k^2)\alpha-3+2k^2\,]\,u^4 \\ &\quad +\, 16\,(1-k^2)\beta\,u^3 \\ &\quad +\, 3k\,[\,\alpha^2+\beta^2+2(1-k^2)\alpha-3+2k^2\,]\,u^2 \\ &\quad +\, k\,[\,\alpha^2+\beta^2-2(1-k^2)\alpha+1-2k^2\,]\,, \end{aligned} \tag{2.17}$$

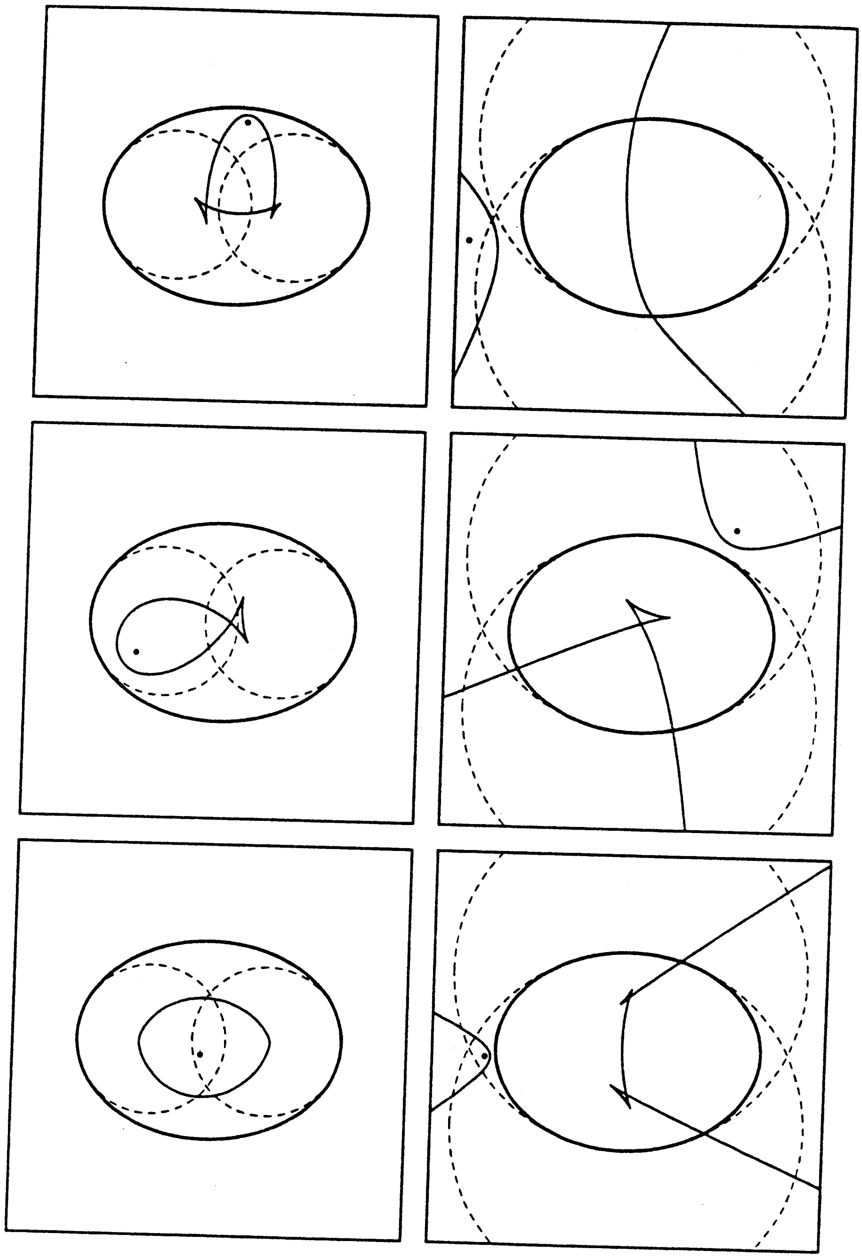


FIGURE 6. Examples of untrimmed point/ellipse bisectors (also shown are circles of curvature to the ellipse at its vertices).

and although this of degree six, it can be shown that there are actually at most *four* distinct real roots. However, the parameter values of the cusps do not (in general) admit a closed–form expression in terms of radicals.

Finally, the self-intersection polynomial $P_i(u)$ for the present example transpires to be the product of two quartic factors

$$\begin{aligned} P_{i,1}(u) &= (\alpha^2+\beta^2+2(1-k^2)\alpha+1-2k^2)u^4 \\ &+ 2(\alpha^2+\beta^2-1)u^2 + \alpha^2+\beta^2-2(1-k^2)\alpha+1-2k^2\,, \\ P_{i,2}(u) &= k(\alpha^2+\beta^2-1)u^4 + 4(1-k^2)\beta u^3 \\ &+ 2k(\alpha^2+\beta^2-3+2k^2)u^2 + 4(1-k^2)\beta u + k(\alpha^2+\beta^2-1)\,. \end{aligned} \tag{2.18}$$

A detailed analysis [8] reveals that $P_{i,1}(u)$ has real roots only when $P_{i,2}(u)$ has none, and vice–versa. In particular, for $P_{i,1}(u)$ or $P_{i,2}(u)$ to have real roots the quantity

$$\Delta = 4(1-k^2)(k^2-k^2\alpha^2-\beta^2) \tag{2.19}$$

must be positive or negative, respectively. The number of these roots is either zero, two, or four (corresponding to zero, one, or two self–intersections) and is determined by the location of $\mathbf{p}$ with respect to the circles of curvature

$$\begin{aligned} C_0(x,y) &= (x-1+k^2)^2+y^2-k^4 = 0\,, \\ C_\infty(x,y) &= (x+1-k^2)^2+y^2-k^4 = 0\,, \\ C_{+1}(x,y) &= x^2+(y^2-k+1/k)^2-1/k^2 = 0\,, \\ C_{-1}(x,y) &= x^2+(y^2+k-1/k)^2-1/k^2 = 0 \end{aligned} \tag{2.20}$$

of the ellipse at its vertices ($u = 0, \infty$ and $u = \pm 1$).

When $\mathbf{p}$ is *inside* the ellipse, $\mathbf{b}(u)$ has two, one, or zero self–intersections, according to whether $\mathbf{p}$ lies inside neither, just one, or both of the circles of curvature to the ellipse at the vertices on its major axis (the last case being possible only if the circles of curvature actually overlap). If $\mathbf{p}$ is *outside* the ellipse, $\mathbf{b}(u)$ has zero, one, or two self–intersections when $\mathbf{p}$ lies inside neither, just one, or both the circles of curvature to the ellipse at the vertices on its minor axis, respectively. In the latter case, of course, $\mathbf{b}(u)$ also has points at infinity identified by (2.16). This behavior is evident in Figure 6.

When $\Delta > 0$, the parameter values that identify self–intersections of the untrimmed bisector are given explicitly by the formula

$$u = \pm\sqrt{\frac{1-\alpha^2-\beta^2\pm\sqrt{\Delta}}{\alpha^2+\beta^2+2(1-k^2)\alpha+1-2k^2}}\,, \tag{2.21}$$

which defines zero, one, or two pairs of real values of equal magnitude and opposite sign, according to the criteria enumerated above. If $\Delta < 0$, the

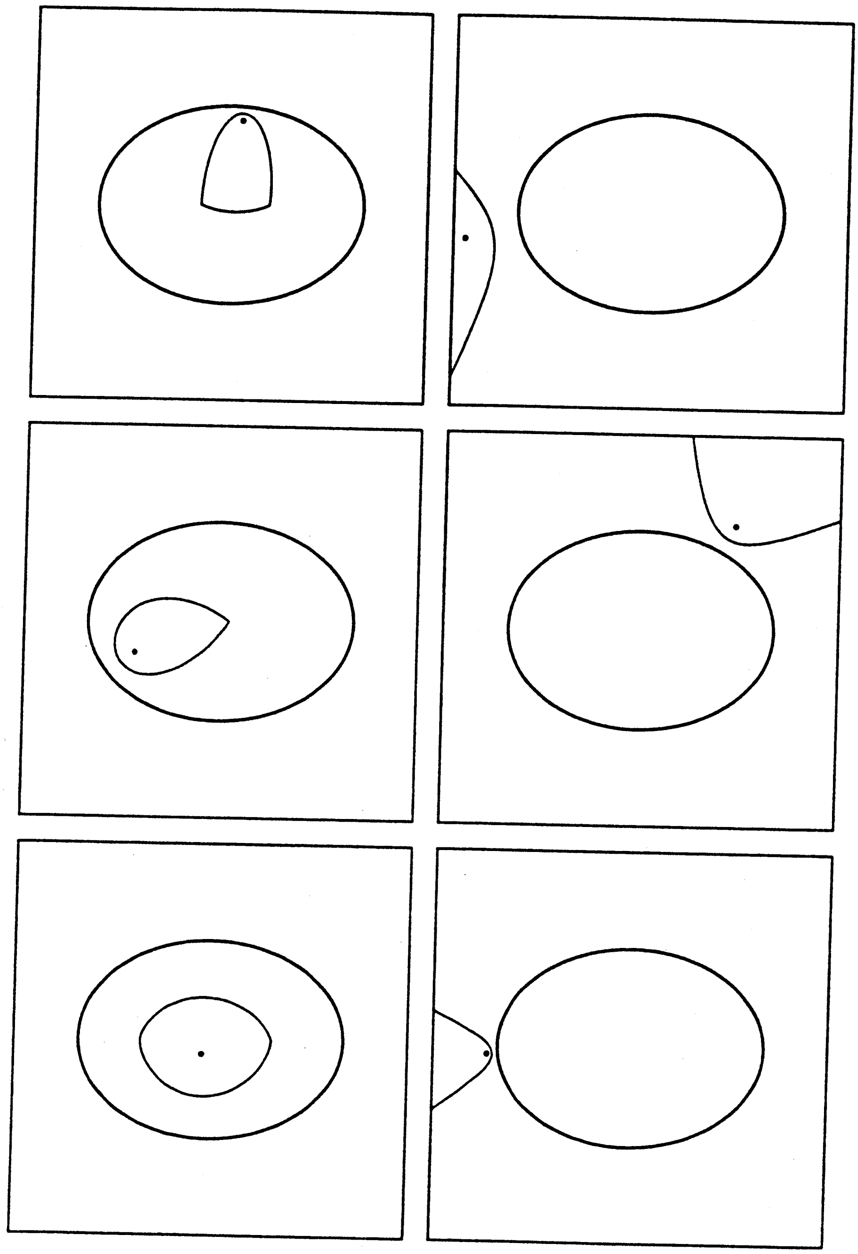


FIGURE 7. The "true" (trimmed) point/ellipse bisectors corresponding to the cases illustrated in Figure 6.

self–intersections can be found by "completing the square" in the polynomial $P_{i,2}(u)$. This gives the two quadratic equations

$$k(\alpha^2+\beta^2-1)\,u^2 \;+\; [\,2(1-k^2)\beta \pm \sqrt{-\Delta}\,]\,u \;+\; k(\alpha^2+\beta^2-1) \;=\; 0 \quad (2.22)$$

whose real solutions are readily determined. Examples of true point/ellipse bisectors obtained after trimming are shown in Figure 7.

3 Curve/curve bisectors as envelopes

The bisector of two regular plane parametric curves, $\mathbf{r}(u)$ and $\mathbf{s}(v)$, is the point locus defined by

$$\mathcal{B} \;=\; \{\,\mathbf{q}\mid \mathrm{dist}(\mathbf{q},\mathbf{r}(u)) = \mathrm{dist}(\mathbf{q},\mathbf{s}(v))\,\}\,. \qquad (3.1)$$

Given a starting point $\mathbf{q}_0$ on $\mathcal{B}$ and the corresponding parameter values u_0 and v_0 that identify the feet of the perpendiculars from $\mathbf{q}_0$ to $\mathbf{r}(u)$ and $\mathbf{s}(v)$ at which $\mathrm{dist}(\mathbf{q}_0,\mathbf{r}(u))$ and $\mathrm{dist}(\mathbf{q}_0,\mathbf{s}(v))$ are realized, one can attempt to trace $\mathcal{B}$ numerically by discrete steps. To first order, the increment $\Delta\mathbf{q}$ to $\mathbf{q}$ at each step lies along the tangent to $\mathcal{B}$, which bisects the angle between the current perpendiculars from $\mathbf{q}$ to the two curves.

For the case of polynomial or rational curves, a sophisticated and more efficient approach along these lines could invoke continuation methods [20] to track the motion of the feet of the perpendiculars from $\mathbf{q}$ to $\mathbf{r}(u)$ and $\mathbf{s}(v)$ as $\mathbf{q}$ traverses $\mathcal{B}$, rather than resorting to explicit root–solving at each step.

However, to maintain a reasonable discretization error with a first–order method would require minuscule steps, and the formulation of second or higher–order expansions along $\mathcal{B}$ is not trivial. Furthermore, detecting and accommodating tangent discontinuities along $\mathcal{B}$ due to sudden jumps in the feet of the appropriate perpendiculars from $\mathbf{q}$ to $\mathbf{r}(u)$ and $\mathbf{s}(v)$ is a delicate task, fraught with difficulties in the context of such numerical curve–tracing procedures (similar problems arise at points where $\mathbf{r}(u)$ and $\mathbf{s}(v)$ cross).

3.1 Envelopes of families of point/curve bisectors

Let $\mathcal{C}(\lambda)$ be a one–parameter family of plane curves that depend continuously on the variable λ. There are two equivalent definitions for the "envelope" of such a family of curves (see [1, Chapter 4] or [14, Chapter 5]):

- The envelope $\mathcal{E}$ is a plane curve that is tangent, at each of its points, to *some* curve in the family $\mathcal{C}(\lambda)$.

- The envelope $\mathcal{E}$ is the locus, as λ varies, of the intersection points of "neighboring" curves $\mathcal{C}(\lambda)$ and $\mathcal{C}(\lambda+\Delta\lambda)$, in the limit $\Delta\lambda \to 0$.

See also Chapter 5 of [3] for an alternate view of the envelope $\mathcal{E}$ of $\mathcal{C}(\lambda)$.

Proposition 3.1 *Let* $\mathbf{b}_\lambda(u)$ *denote the point/curve bisector for the discrete point* $v = \lambda$ *on* $\mathbf{s}(v)$ *and the curve* $\mathbf{r}(u)$ *taken in its entirety. Then the bisector* $\mathcal{B}$ *of the two curves* $\mathbf{r}(u)$ *and* $\mathbf{s}(v)$ *is a subset of the envelope* $\mathcal{E}$ *of the family* $\mathcal{C}(\lambda) = \mathbf{b}_\lambda(u)$ *of such point/curve bisectors.*

Proof (sketch): Let $\mathcal{C}(\lambda)$, $\mathcal{C}(\lambda + \Delta\lambda)$ be the bisectors of the discrete points $\mathbf{s}(\lambda)$, $\mathbf{s}(\lambda + \Delta\lambda)$ on $\mathbf{s}(v)$ and the entire curve $\mathbf{r}(u)$, respectively. If $\mathbf{q}$ is a point of intersection of $\mathcal{C}(\lambda)$ and $\mathcal{C}(\lambda + \Delta\lambda)$, then by definition we have

$$|\mathbf{q} - \mathbf{s}(\lambda)| = |\mathbf{q} - \mathbf{s}(\lambda + \Delta\lambda)| = \mathrm{dist}(\mathbf{q}, \mathbf{r}(u)). \tag{3.2}$$

When $\Delta\lambda \to 0$ the above condition guarantees that, as λ varies, $\mathbf{q}$ moves so as to remain equidistant from the entire curve $\mathbf{r}(u)$ and some neighborhood of the point $v = \lambda$ on the curve $\mathbf{s}(v)$. The motion of $\mathbf{q}$ as λ varies will then yield an arc of the true bisector $\mathcal{B}$ of $\mathbf{r}(u)$ and $\mathbf{s}(v)$ if and only if no other point of the curve $\mathbf{s}(v)$ is closer to $\mathbf{q}$ than $\mathbf{s}(\lambda)$, *i.e.*, $|\mathbf{q} - \mathbf{s}(\lambda)| = \mathrm{dist}(\mathbf{q}, \mathbf{s}(v))$. If the intersection $\mathbf{q}$ is such that $|\mathbf{q} - \mathbf{s}(\lambda)| > \mathrm{dist}(\mathbf{q}, \mathbf{s}(v))$, its motion yields an "extraneous" arc of the envelope $\mathcal{E}$ that does not belong to $\mathcal{B}$. ∎

Figure 8 illustrates the above ideas in the context of two ellipses, where a family of point/curve bisectors has been computed by taking a sequence of discrete points along one of the ellipses. The envelope $\mathcal{E}$ stands out as the locus along which the individual point/curve bisectors "concentrate." Note also that, in both examples, there are portions of the envelope that are clearly ***not*** segments of the true bisector $\mathcal{B}$ of the two ellipses.

3.2 Tracing the envelope

We know that, for each λ, the bisector $\mathbf{b}_\lambda(u)$ of the point $\mathbf{s}(\lambda)$ and the curve $\mathbf{r}(u)$ consists of a finite number of parametric subsegments of a single rational curve. These arcs are connected at their endpoints so as to bound a convex, simply–connected region. Mapping each arc of $\mathbf{b}_\lambda(u)$ to the unit interval $u \in [0,1]$, we may represent them as rational Bézier curves, and standard techniques [5] for the subdivision of their control polygons then yield a convergent sequence of polygonal approximations to $\mathbf{b}_\lambda(u)$. Intersecting the polygonal approximations to $\mathbf{b}_\lambda(u)$ and $\mathbf{b}_{\lambda+\Delta\lambda}(u)$, for small increments $\Delta\lambda$, offers a simple and relatively "robust" — though rather brutish — means of approximating (a superset of) the bisector $\mathcal{B}$ of $\mathbf{r}(u)$ and $\mathbf{s}(v)$.

However, with a little analysis we can do much better. The derivation of envelope equations is usually discussed [1, 14] in the context of families of curves defined by an *implicit* (algebraic) equation

$$f(x, y, \lambda) = 0. \tag{3.3}$$

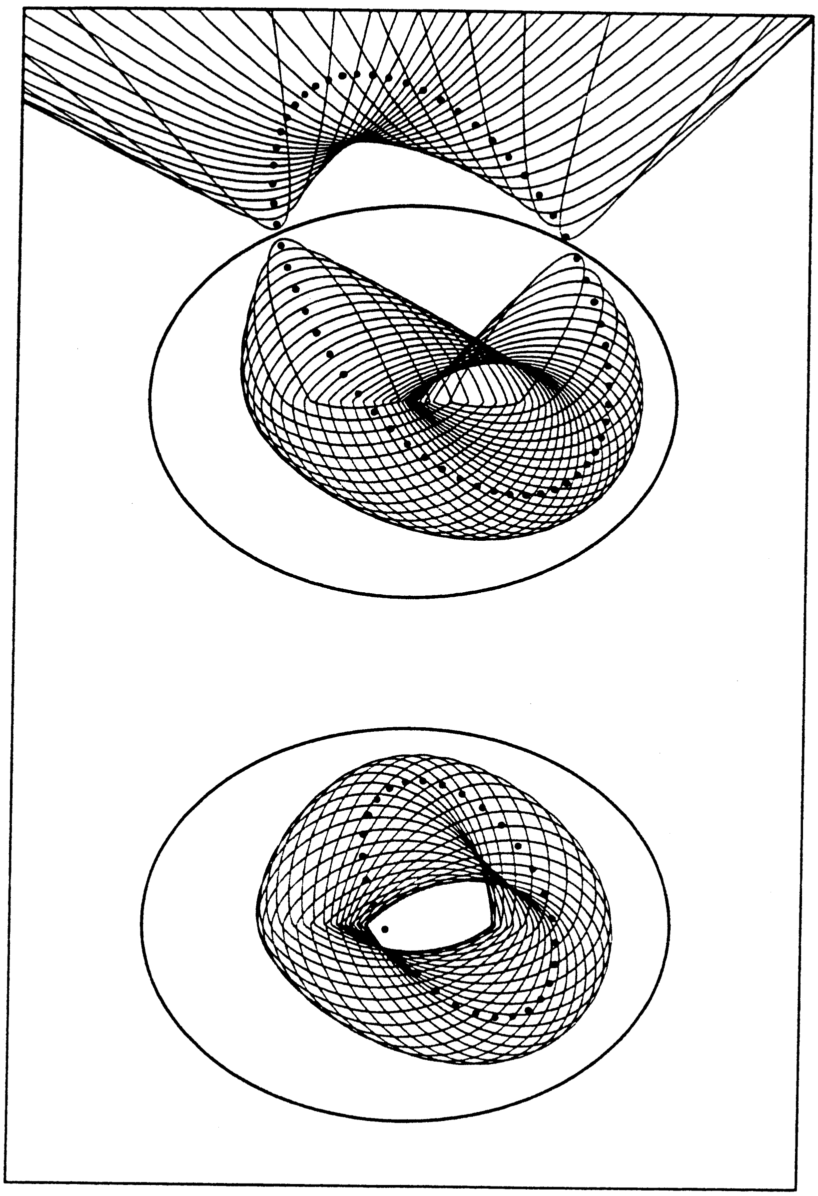


FIGURE 8. Examples of the bisector of two ellipses as the envelope of a family of point/ellipse bisectors.

A necessary condition for the point (x, y) to lie on the envelope $\mathcal{E}$ of (3.3) is that, for *some* λ, it should satisfy

$$f(x,y,\lambda) = \frac{\partial f}{\partial \lambda}(x,y,\lambda) = 0. \tag{3.4}$$

Thus, eliminating λ between the two equations $f = 0$ and $\partial f/\partial\lambda = 0$, *i.e.*, computing their "resultant" [23] with respect to λ:

$$e(x,y) = \text{Resultant}_\lambda \left(f, \frac{\partial f}{\partial \lambda} \right), \tag{3.5}$$

defines an algebraic curve $e(x, y) = 0$ that contains the envelope $\mathcal{E}$. However, factors extraneous to the envelope can arise under various circumstances. For example, if (3.3) degenerates such that $f(x, y, \lambda_0) \equiv 0$ at $\lambda = \lambda_0$, then $\partial f(x, y, \lambda_0)/\partial\lambda$ will appear as a factor in $e(x, y)$ — and vice–versa. Similarly, if f and $\partial f/\partial\lambda$ have a non–constant common factor $\gamma(x, y)$ at $\lambda = \lambda_0$, this will also appear in $e(x, y)$. Finally, if (3.3) exhibits a *locus of singular points* (*i.e.*, a curve $\sigma(x, y) = 0$ such that, as λ varies, $f = \partial f/\partial x = \partial f/\partial y = 0$ at each point of this curve) then $\sigma(x, y)$ will be a component of $e(x, y)$.

Equation (3.4) indicates that the discrete points that each curve λ of the family contributes to the envelope can be regarded as the intersections of $f(x, y, \lambda) = 0$ with the curve $\partial f(x, y, \lambda)/\partial\lambda = 0$. The above approach could, in principle, be applied to the envelopes of families of point/curve bisectors, by first "implicitizing" [22] the rational parametric form $\mathbf{b}_\lambda(u)$ (of course, the resulting equation $b(x, y, \lambda) = 0$ would represent the *untrimmed* bisectors). However, even in simple cases the envelope equations $e(x, y) = 0$ generated by this approach are unwieldy and of little practical use.

We prefer to work directly with the parametric representation $\mathbf{b}_\lambda(u)$, and we now develop a criterion analogous to (3.4) that is appropriate to this form.

Theorem 3.1 *Let* $\mathbf{b}_\lambda(u)$ *be the bisector of the point* $v = \lambda$ *on* $\mathbf{s}(v)$ *and the curve* $\mathbf{r}(u)$. *Then, for each* λ, *the smooth points of* $\mathbf{b}_\lambda(u)$ *that contribute to the envelope* $\mathcal{E}$ *of the family of point/curve bisectors* $\mathcal{C}(\lambda) = \mathbf{b}_\lambda(u)$ *are identified by the parameter values* u *for which*

$$\frac{\partial \mathbf{b}_\lambda}{\partial \lambda}(u) = \mathbf{0}. \tag{3.6}$$

Proof (sketch): The intersections of "neighboring" members λ and $\lambda + \Delta\lambda$ of the family of point/curve bisectors can be identified with parameter values u for which there exist increments Δu such that

$$\mathbf{b}_\lambda(u) = \mathbf{b}_{\lambda+\Delta\lambda}(u + \Delta u). \tag{3.7}$$

Assuming $\mathbf{b}_\lambda(u)$ is a smooth point (*i.e.*, not the juncture of two distinct arcs belonging to the point/curve bisector) we can expand the above to obtain

$$\mathbf{b}_\lambda'(u)\,\Delta u + \frac{\partial \mathbf{b}_\lambda}{\partial \lambda}(u)\,\Delta\lambda + \cdots = \mathbf{0}, \tag{3.8}$$

where, as usual, primes denote differentiation with respect to u. As this is a *vector* equation, it can be satisfied (to first order) only when the derivatives $\mathbf{b}_\lambda'$ and $\partial\mathbf{b}_\lambda/\partial\lambda$ are parallel — or one of them vanishes. Writing

$$\mathbf{b}_\lambda(u) = \mathbf{r}(u) + d_\lambda(u)\mathbf{n}(u), \tag{3.9}$$

where $d_\lambda(u)$ is defined by substituting $\mathbf{s}(\lambda)$ for $\mathbf{p}$ in (2.3), we differentiate the above and invoke the Frenet relations [18] to obtain

$$\mathbf{b}_\lambda' = |\mathbf{r}'|(1+\kappa d_\lambda)\,\mathbf{t} + d_\lambda'\,\mathbf{n} \qquad \text{and} \qquad \frac{\partial \mathbf{b}_\lambda}{\partial\lambda} = \frac{\partial d_\lambda}{\partial\lambda}\,\mathbf{n}. \tag{3.10}$$

Here $\mathbf{t}$, $\mathbf{n}$, and κ are the tangent, normal, and curvature of $\mathbf{r}(u)$, and $|\mathbf{r}'| \neq 0$ for a regular curve. Thus we must have either $\kappa = -1/d_\lambda$ or $|d_\lambda'| \to \infty$ for the derivatives (3.10) to be parallel if they are non–zero. But the first condition identifies *cusps* on the untrimmed bisector of $\mathbf{s}(\lambda)$ and $\mathbf{r}(u)$, which cannot belong to the true bisector (the latter must be convex). Similarly, the second condition is discounted since it identifies *points at infinity* on $\mathbf{b}_\lambda(u)$.

Hence, affine smooth points of $\mathbf{b}_\lambda(u)$ that contribute to the envelope must satisfy $\mathbf{b}_\lambda' = \mathbf{0}$ or $\partial\mathbf{b}_\lambda/\partial\lambda = \mathbf{0}$. Again, the first case can be dropped since it identifies cusps, and we are left with the stated condition (3.6). Equation (3.8) then gives $\Delta u = 0$, *i.e.*, when (3.6) is satisfied at $u = u_*$, say, then to first order the intersection of the neighboring members $\mathbf{b}_\lambda(u)$ and $\mathbf{b}_{\lambda+\Delta\lambda}(u)$ corresponds to the *same* parameter value u_* on these two curves. ∎

Note that, unlike the numerical schemes mentioned above, condition (3.6) gives points that lie *exactly* on the envelope $\mathcal{E}$ of the family $\mathcal{C}(\lambda) = \mathbf{b}_\lambda(u)$. From (3.10) we see that condition (3.6) amounts to the requirement that the partial derivative of the displacement function

$$d_\lambda(u) = \frac{|\,\mathbf{s}(\lambda) - \mathbf{r}(u)\,|^2}{2\,(\mathbf{s}(\lambda) - \mathbf{r}(u))\cdot\mathbf{n}(u)} \tag{3.11}$$

with respect to λ should vanish. This occurs when

$$2\,(\mathbf{s}(\lambda) - \mathbf{r}(u))\cdot\mathbf{n}(u)\;(\mathbf{s}(\lambda) - \mathbf{r}(u))\cdot\frac{\partial\mathbf{s}}{\partial\lambda} = |\,\mathbf{s}(\lambda) - \mathbf{r}(u)\,|^2\,\mathbf{n}(u)\cdot\frac{\partial\mathbf{s}}{\partial\lambda} \tag{3.12}$$

which, for each λ, amounts to a polynomial equation in u whose real roots on the domain of definition of the point/curve bisector $\mathbf{b}_\lambda(u)$ are desired. We now give a geometric interpretation of the above equation:

Proposition 3.2 *For each λ, the roots of equation (3.12) identify the points of $\mathbf{b}_\lambda(u)$ that lie on the normal line to the curve $\mathbf{s}(v)$ at $v = \lambda$.*

Proof : We divide (3.12) through by $2\,(\mathbf{s}(\lambda) - \mathbf{r}(u)) \cdot \mathbf{n}(u)$ and re-write it as

$$\left[\frac{|\,\mathbf{s}(\lambda) - \mathbf{r}(u)\,|^2}{2\,(\mathbf{s}(\lambda) - \mathbf{r}(u)) \cdot \mathbf{n}(u)}\,\mathbf{n}(u) \;-\; (\mathbf{s}(\lambda) - \mathbf{r}(u)) \right] \cdot \frac{\partial \mathbf{s}}{\partial \lambda} = 0\,. \qquad (3.13)$$

The scalar quantity multiplying $\mathbf{n}(u)$ above is just the displacement function $d_\lambda(u)$ given by (3.11), and since $\mathbf{b}_\lambda(u) = \mathbf{r}(u) + d_\lambda(u)\mathbf{n}(u)$ we obtain

$$[\,\mathbf{b}_\lambda(u) - \mathbf{s}(\lambda)\,] \cdot \frac{\partial \mathbf{s}}{\partial \lambda} = 0\,. \qquad (3.14)$$

Now $\partial\mathbf{s}/\partial\lambda$ ($\neq \mathbf{0}$) gives the tangent direction to the (regular) curve $\mathbf{s}(v)$ at $v = \lambda$, so the value of u must be such that the point $\mathbf{b}_\lambda(u)$ lies on the normal line to $\mathbf{s}(v)$ at $v = \lambda$ to satisfy the above. ■

Thus, Proposition 3.2 allows to easily pick off the (smooth) points on each member of the family $\mathcal{C}(\lambda) = \mathbf{b}_\lambda(u)$ of point/curve bisectors that contribute to the envelope $\mathcal{E}$ of this family: simply intersect $\mathbf{b}_\lambda(u)$ with the normal line to $\mathbf{s}(v)$ at $v = \lambda$. By the fact that each point/curve bisector is convex (see Remark 2.7), $\mathbf{b}_\lambda(u)$ contributes at most two affine points to $\mathcal{E}$ for each λ.

Remark 3.1 If we could write the roots of (3.12) as *explicit* functions $u(\lambda)$ of λ, these could be substituted into the point/curve bisector equation to give a (local) parameterization $\mathbf{b}_\lambda(u(\lambda))$ of the bisector $\mathcal{B}$ of $\mathbf{r}(u)$ and $\mathbf{s}(v)$. However, this is impossible even in the simplest non-trivial cases. If $\mathbf{r}(u)$ is a polynomial or rational curve of degree ≥ 2, for example, equation (3.12) is, respectively, of degree $3n - 1 \geq 5$ or $4n - 2 \geq 6$ in u, and cannot be solved in terms of radicals. Thus, the curve/curve bisector $\mathcal{B}$ has no "simple" parameterization, and must be traced by some numerical scheme.

Finally, we must address the possible contributions of non-smooth points on the individual point/curve bisectors $\mathbf{b}_\lambda(u)$ to their overall envelope. These are the endpoints of the various parametric subsegments that constitute the true point/curve bisector, which will generally meet with only C^0 continuity. A simple example illustrates that one cannot afford to simply ignore such points: when a polygon moves in the plane, the envelope that bounds the area it sweeps out is generated *entirely* by non-smooth points (the vertices) if the sense of motion is never parallel to any side of the polygon.

We will content ourselves here with the observation that, for each λ, one could simply include *all* non-smooth points of $\mathbf{b}_\lambda(u)$ as candidate points for the curve/curve bisector $\mathcal{B}$. Typically, there are only a few such points, and their location is known without further calculation. Since the smooth

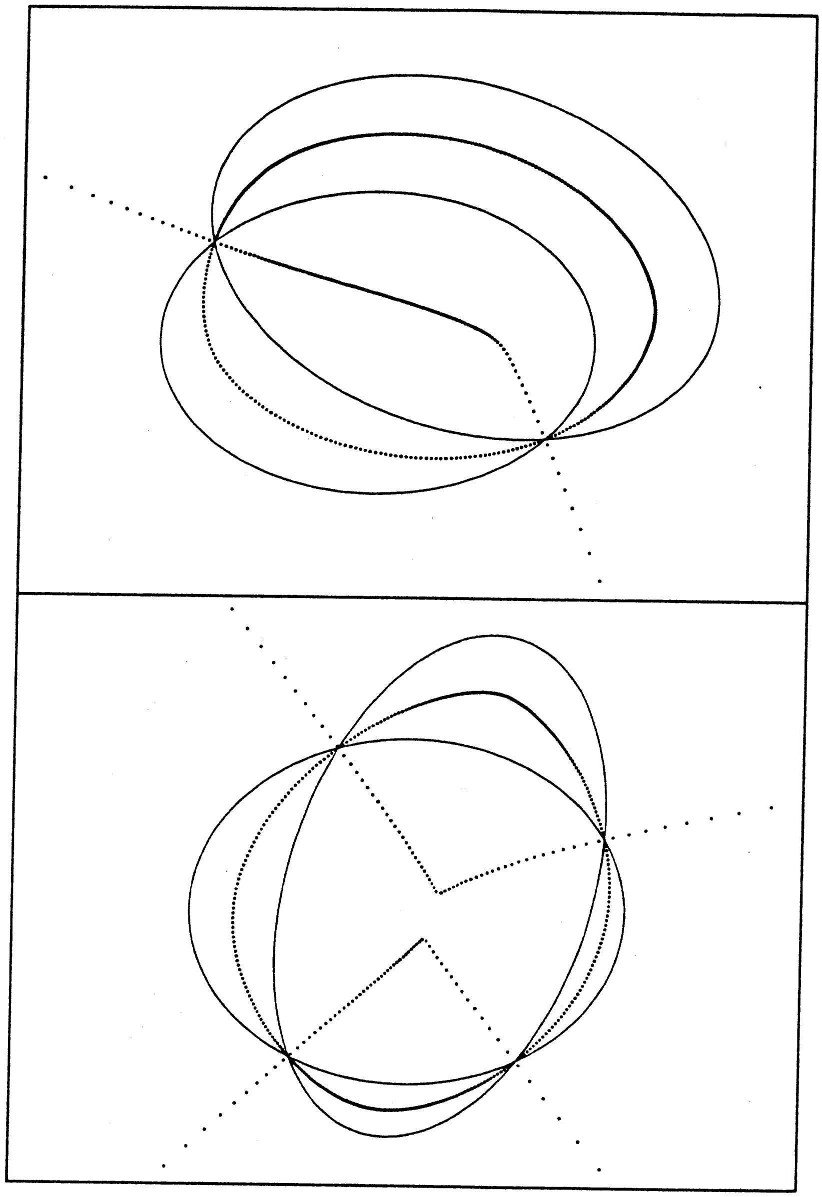


FIGURE 9. Sampling of curve/curve bisectors.

points of each $\mathbf{b}_\lambda(u)$ that contribute to their envelope $\mathcal{E}$ must be validated for membership in $\mathcal{B}$, by testing for equality of their distances from $\mathbf{r}(u)$ and $\mathbf{s}(v)$, one might as well test all the non-smooth points too.

Figure 9 shows examples of curve/curve bisectors that have been sampled using the methods described above. Note that the bisector has two branches passing through any intersection of the given curves; the bisector tangents at such points bisect the angles between the tangents to the given curves.

4 Concluding remarks

We now conclude by briefly summarizing all the above results:

- The bisector $\mathbf{b}(u)$ of a point $\mathbf{p}$ and a polynomial or rational parametric curve $\mathbf{r}(u)$ of degree n consists of a finite number of subsegments of a single rational curve of degree $3n-1$ or $4n-2$, respectively, that bound a convex, simply-connected region of the plane. Given a robust polynomial root-finder, it can be computed in an algorithmic manner.

- The bisector $\mathcal{B}$ of two parametric curves $\mathbf{r}(u)$ and $\mathbf{s}(v)$ is a subset of the envelope $\mathcal{E}$ of the family of point/curve bisectors $\mathbf{b}_\lambda(u)$ that correspond to the discrete point $v = \lambda$ on $\mathbf{s}(v)$ and the entire curve $\mathbf{r}(u)$.

- The smooth points of each $\mathbf{b}_\lambda(u)$ that lie on the envelope $\mathcal{E}$ are identified by the intersections of the normal line to $\mathbf{s}(v)$ at $v = \lambda$ with $\mathbf{b}_\lambda(u)$. All non-smooth points of each $\mathbf{b}_\lambda(u)$ can be regarded as candidates for $\mathcal{E}$. Increasing λ by small increments, we can test these points for membership in $\mathcal{B} \subset \mathcal{E}$ by comparing their distances to $\mathbf{r}(u)$ and $\mathbf{s}(v)$, and thus trace out the bisector of the two curves.

Some refinements to the method outlined above are needed to yield a practical scheme for tracing curve/curve bisectors. First, if we systematically increase λ, the envelope points are not generated with proper "ordering" along the bisector. Thus, a method for sorting them *a posteriori* or detecting breakdown of ordering during run-time is required. Second, the spacing of the envelope points can be very uneven if uniform increments in λ are used. A scheme that adaptively selects the λ increments to give a more even spacing is desirable. We hope to address these problems in due course.

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