The bisector of a point and a plane parametric curve

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Abstract

The *bisector* of a fixed point $\mathbf{p}$ and a smooth plane curve C — *i.e.*, the locus traced by a point that remains equidistant with respect to $\mathbf{p}$ and C — is investigated in the case that C admits a regular polynomial or rational parameterization. It is shown that the bisector may be regarded as (a subset of) a "variable–distance" offset curve to C which has the attractive property, unlike fixed–distance offsets, of being *generically* a rational curve. This "untrimmed bisector" usually exhibits irregular points and self-intersections similar in nature to those seen on fixed–distance offsets. A *trimming procedure*, which identifies the parametric subsegments of this curve that constitute the true bisector, is described in detail. The bisector of the point $\mathbf{p}$ and any finite segment of the curve C is also discussed.

1 Introduction

In descriptive geometry [3] the parabola is characterized as the locus traced by a point that remains equidistant with respect to a fixed point **p** (the *focus*) and a given straight line L (the *directrix*). Thus, points of the plane that lie to one side of the parabola are closer to **p** than to L, while those that lie on the other side are closer to L than to **p**. In this sense, the parabola may be regarded as the "bisector" of the point **p** and the line L.

If we substitute a smooth plane curve C in place of the straight line L, the bisector locus is of a more subtle nature. This paper is concerned with investigating the geometric properties of such loci, and formulating tractable representations for them. Point/curve bisectors arise in a variety of geometric "reasoning" and geometric decomposition problems (*e.g.*, planning paths of maximum clearance in robotics, or computing Voronoi diagrams for areas with curvilinear boundaries). Although they are much simpler than other loci — line/curve and curve/curve bisectors — that arise in these contexts, no systematic analysis of the properties of point/curve bisectors is currently available in the literature.

However, there has been considerable interest recently in the general topic of bisectors. They play a key rôle in computing the *medial axis transform* or "skeleton" of planar shapes (see, for example, Bookstein [1] and Lee [19]). Yap [27] discusses the bisectors of points, lines, and circles in the context of Voronoi diagrams. Also in the context of Voronoi diagrams, Held [11] treats the construction of bisectors in numerical–control machining applications. Yap and Alt [28] analyze the complexity of the bisector computation for two algebraic curves, quoting an upper bound of $16m^6$ on the degree of the bisector for curves of degree m. Nackman and Srinivasan [20] discuss generic properties of the bisector of two linearly–separable sets of arbitrary dimension, from the perspective of point–set topology.

Hoffmann and Vermeer [14] develop systems of equations that define "equal–distance" curves and surfaces (another term for bisectors, viewed as offsets from given curves/surfaces). Voronoi surfaces — *i.e.*, the bisectors of two given surfaces — are also discussed by Hoffmann [13] and Dutta and Hoffmann [4]. Finally, we note that the notion of the "offset" to a given curve (or surface) is closely related to that of bisectors; this relationship will be directly exploited in the development of this paper. A detailed discussion of offset curves is given by Farouki and Neff [6, 7].

The organization of this paper is as follows. Section 1 outlines some basic properties of regular parametric curves, the point/curve distance function, and point/curve bisectors. After reviewing the notion of an offset curve in Section 2.1, we show in Section 2.2 how to represent the untrimmed bisector, which is a superset of the true bisector (see Section 2.4), as a "variable-distance" offset. Section 2.3 discusses the irregular points that may arise on this locus. The trimming procedure, whereby the true bisector is obtained from the untrimmed bisector, is the subject of Section 3. A key step in the trimming process is the computation of self-intersections of the untrimmed bisector (see Section 3.3). Finally, Section 4 summarizes our main results.

1.1 Regular parametric curves

We shall focus here on the important case where the curve C is described parametrically, $\mathbf{r}(u) = \{x(u), y(u)\}$, having derivatives

$$\mathbf{r}'(u) = \{x'(u), y'(u)\}\,, \quad \mathbf{r}''(u) = \{x''(u), y''(u)\}\,, \quad \ldots \text{etc.} \tag{1}$$

continuous to at least third order for all $u \in I$, where I denotes some finite, semi-infinite, or infinite parameter domain of interest. We assume that the parameterization of $\mathbf{r}(u)$ is *proper*, *i.e.*, there is a one-to-one correspondence between parameter values u and points (x, y) of the curve locus — except, possibly, for finitely many instances where $\mathbf{r}(u)$ crosses itself.

Since improper parameterizations arise rather infrequently in practice, and identifying them is not in general a straightforward matter (see [22, 23]), we shall not dwell on this issue. However, we do need to impose an additional constraint on the parameterization of $\mathbf{r}(u)$ — namely, that it be *regular* over the parameter domain of interest:

Definition 1.1 *The parametric speed of* $\mathbf{r}(u) = \{x(u), y(u)\}$ *is the function*

$$\sigma(u) = \sqrt{x'^2(u) + y'^2(u)} \tag{2}$$

of the parameter u*, and the curve is said to have a regular parameterization on the interval* I *if and only if* $\sigma(u) \neq 0$ *for all* $u \in I$*.*

It should be noted that any polynomial or rational curve that has an irregular parameterization will not, in general, exhibit a smooth locus: the points where $\sigma(u) = 0$ correspond to *cusps* (sudden tangent reversals) or, exceptionally, discontinuities in higher–order differential characteristics [10].

We denote by $|\mathbf{v}|$ the Euclidean norm $\sqrt{v_x^2 + v_y^2}$ of a vector $\mathbf{v} = (v_x, v_y)$. Thus, we shall also write $|\mathbf{r}'(u)|$ for the parametric speed of $\mathbf{r}(u)$, to suit the context. Since we are concerned solely with *real* functions $x(u)$ and $y(u)$ of a *real* parameter u, we note that $\sigma(u) = 0 \iff x'(u) = y'(u) = 0$.

Although much of the ensuing discussion holds for any regular parametric curve, we shall deal primarily with the two functional forms encountered most often in practice: the *polynomial* curve $\mathbf{r}(u) = \{X(u), Y(u)\}$ of degree n defined by

$$X(u) = \sum_{k=0}^{n} a_k u^k\,, \quad Y(u) = \sum_{k=0}^{n} b_k u^k\,, \tag{3}$$

the coefficients $\{a_k, b_k\}$ being real numbers that satisfy $a_n^2 + b_n^2 \neq 0$, and the *rational* curve $\mathbf{r}(u) = \{X(u)/W(u), Y(u)/W(u)\}$ of degree n, where

$$X(u) = \sum_{k=0}^{n} a_k u^k\,, \quad Y(u) = \sum_{k=0}^{n} b_k u^k\,, \quad W(u) = \sum_{k=0}^{n} c_k u^k\,, \tag{4}$$

the coefficients $\{a_k, b_k, c_k\}$ being again real numbers with either $a_n^2 + b_n^2 \neq 0$ or $c_n \neq 0$. In (4) we also assume that there are no factors common to *all three* of the polynomials X, Y, W, *i.e.*, that $\text{GCD}(X, Y, W) = \text{constant}$. ($\text{GCD}(\cdots)$ denotes the "greatest common divisor" of the indicated set of polynomials, as determined by one or more applications of Euclid's algorithm [25].)

Since polynomial curves are a special class of rational curves with $W(u) =$ constant, when we speak specifically of rational curves it will be understood implicitly that $W(u) \neq$ constant. Roots of the polynomial $W(u)$ correspond to "points at infinity" on the rational curve (4). In most applications, we are concerned only with the *affine* part of a rational curve, *i.e.*, its locus for all parameter values u such that $W(u) \neq 0$.

Remark 1.1 A sufficient and necessary condition for the polynomial curve (3) to have a regular parameterization is that

$$\text{GCD}(X', Y') = \text{constant}\,, \tag{5}$$

while in the case of the rational curve (4) we require

$$\frac{\text{GCD}(WX' - W'X, WY' - W'Y)}{\text{GCD}(W, W')} = \text{constant} \tag{6}$$

for the affine locus to have a regular parameterization.

Example 1.1 We have already noted the simple nature of the bisector of a point and a straight line. Another common case that yields an "elementary" bisector arises if we take the curve C to be a *circle*. Assume, without loss of generality, that C is of unit radius and centered on the origin, and let the given point be $\mathbf{p} = (\alpha, \beta)$. Then the bisector is evidently the locus of points (x, y) that satisfy

$$\left| \sqrt{x^2 + y^2} - 1 \right| = \sqrt{(x-\alpha)^2 + (y-\beta)^2}\,, \tag{7}$$

where the left– and right–hand sides represent the distance of the variable point (x, y) from the circle C and the fixed point $\mathbf{p}$, respectively. Squaring twice to clear radicals, we see that the bisector has the implicit equation

$$\begin{aligned} &(1-\alpha^2)\,x^2 + (1-\beta^2)\,y^2 - 2\alpha\beta\,xy \\ &\quad + (\alpha^2+\beta^2-1)\,(\alpha x + \beta y) - \tfrac{1}{4}(\alpha^2+\beta^2-1)^2 = 0\,, \end{aligned} \tag{8}$$

which clearly represents a conic section. The nature of this conic may be determined [5] by inspecting the signs of the invariants

$$k = (1-\alpha^2)(1-\beta^2) - \alpha^2\beta^2\,, \tag{9}$$

which simplifies to $k = 1 - (\alpha^2 + \beta^2)$, and

$$D = \begin{vmatrix} 1-\alpha^2 & -\alpha\beta & \frac{1}{2}(\alpha^2+\beta^2-1)\alpha \\ -\alpha\beta & 1-\beta^2 & \frac{1}{2}(\alpha^2+\beta^2-1)\beta \\ \frac{1}{2}(\alpha^2+\beta^2-1)\alpha & \frac{1}{2}(\alpha^2+\beta^2-1)\beta & -\frac{1}{4}(\alpha^2+\beta^2-1)^2 \end{vmatrix}\,, \tag{10}$$

which yields $D = -\frac{1}{4}[\,1-(\alpha^2+\beta^2)\,]^2 = -\frac{1}{4}k^2$ on expansion.

When $k = 0$, *i.e.*, the point $\mathbf{p}$ lies *on* the circle C, equation (8) becomes $(\beta x - \alpha y)^2 = 0$, and the bisector degenerates into a straight line — the normal to C at $\mathbf{p}$ — counted twice. Otherwise, the bisector is a non–degenerate conic: an ellipse or (one branch of a) hyperbola according to whether $k > 0$ or $k < 0$, *i.e.*, whether $\mathbf{p}$ lies *inside* or *outside* the circle C. Some examples are illustrated in Figure 1. ■

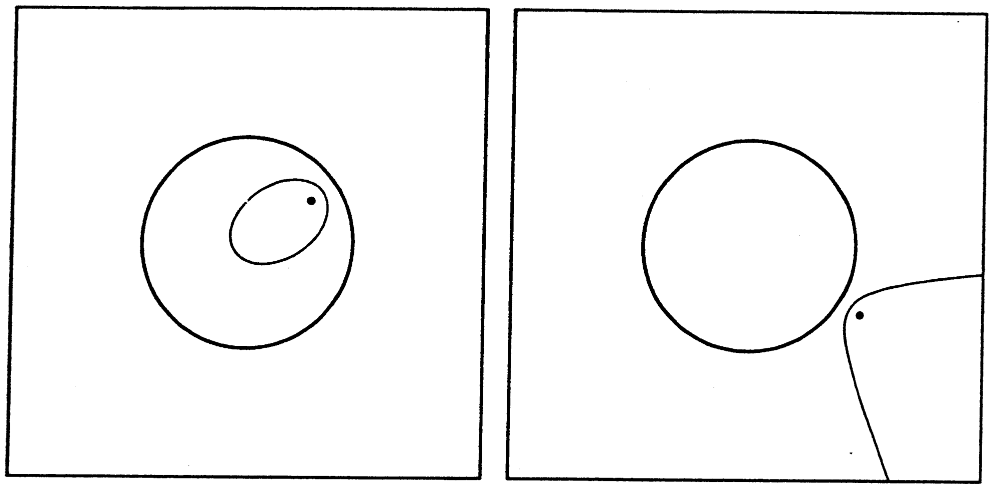

Figure 1: Representative bisectors of a point and a circle.

1.2 The point/curve distance function

In characterizing the parabola as the bisector of a point $\mathbf{p}$ and a straight line L, the meaning of the "distance" of any point from L is clear: it is simply the length of the *unique* perpendicular from the point in question to the straight line L. In substituting a smooth parametric curve C in place of L, we need to generalize this notion of distance (see [16]):

Definition 1.2 *The distance of a point* $\mathbf{q}$ *from a regular parametric curve* $\mathbf{r}(u) = \{x(u), y(u)\}$ *defined on the parameter interval* I *is given by*

$$\operatorname{dist}(\mathbf{q}, \mathbf{r}(u)) = \inf_{u \in I} |\, \mathbf{q} - \mathbf{r}(u) \,|, \tag{11}$$

i.e., it is the greatest lower bound, for all $u \in I$*, on the distance measured between the specified point* $\mathbf{q}$ *and each point* $\mathbf{r}(u)$ *along the curve.*

If $\mathbf{r}(u)$ is a polynomial curve, of course, the bound (11) is always attained at a *finite* parameter value u regardless of whether I has finite or infinite extent. When $\mathbf{r}(u)$ is a rational curve and I is not finite, however, it is possible that (11) may be attained in the limit $|u| \to \infty$ if the degree of $W(u)$ is not less than the greater of the degrees of $X(u)$ and $Y(u)$.

Consider, for example, the point $\mathbf{q} = (-2, 0)$ and the unit circle centered on the origin, $\mathbf{r}(u) = \{ (1-u^2)/(1+u^2), 2u/(1+u^2) \}$ for $u \in (-\infty, +\infty)$. The closest point on $\mathbf{r}(u)$ to $\mathbf{q}$ is clearly $\mathbf{r}(\pm\infty) = (-1, 0)$.

Note that as $|u| \to \infty$ the rational curve (4) converges to an affine point (x_∞, y_∞), where $|x_\infty| = |a_n/c_n|$ and $|y_\infty| = |b_n/c_n|$, if and only if $\deg(W) \geq \max(\deg(X), \deg(Y))$; otherwise it has a point at infinity at infinite values of u. In the former case it is always possible to re-parameterize (4) by a bilinear transformation of the parameter so as to make (x_∞, y_∞) correspond to a *finite* parameter value.

Proposition 1.1 *For the point* $\mathbf{q} = (a, b)$ *and the polynomial curve* $\mathbf{r}(u)$ *given by* (3), *let* $\{u_1, \ldots, u_N\}$ *be the set of distinct odd-multiplicity roots of the polynomial*

$$P_\perp(u) = [\, a - X(u)\,]\, X'(u) + [\, b - Y(u)\,]\, Y'(u) \tag{12}$$

of degree $2n-1$ *on the interior of the interval* I, *augmented by the finite end points, if any, of* I. *Then the distance function* (11) *may be expressed as*

$$\mathrm{dist}(\mathbf{q}, \mathbf{r}(u)) = \min_{k \in \{1,\ldots,N\}} |\, \mathbf{q} - \mathbf{r}(u_k)\,| \,. \tag{13}$$

Proof : On differentiating the expression

$$|\, \mathbf{q} - \mathbf{r}(u)\,|^2 = [\, a - X(u)\,]^2 + [\, b - Y(u)\,]^2 \,, \tag{14}$$

we see that the distance $|\,\mathbf{q} - \mathbf{r}(u)\,|$ will attain a "stationary" value whenever $[\, a - X(u)\,]\, X'(u) + [\, b - Y(u)\,]\, Y'(u) = 0$, *i.e.*, at the roots of the polynomial $P_\perp(u)$. (Note that, by virtue of the constraint (5), this equation will never be satisfied in the degenerate case $X'(u) = Y'(u) = 0$.) Only those roots of $P_\perp(u)$ that are of *odd* multiplicity identify local *extrema* of $|\,\mathbf{q} - \mathbf{r}(u)\,|$, however. To evaluate (11) we must compare the values of $|\,\mathbf{q} - \mathbf{r}(u)\,|$ at each odd root of $P_\perp(u)$ on the parameter interval I and at its end points *if* they are finite (since $|\,\mathbf{q} - \mathbf{r}(u)\,| \to \infty$ for any polynomial curve as $|u| \to \infty$). Then $\mathrm{dist}(\mathbf{q}, \mathbf{r}(u))$ is given by the smallest of these values. ∎

A result analogous to Proposition 1.1 holds for regular rational curves, provided we replace the odd roots of the polynomial (12) by those of

$$\begin{aligned} P_\perp(u) &= [\, aW(u) - X(u)\,][\, W(u)X'(u) - W'(u)X(u)\,] \\ &+ [\, bW(u) - Y(u)\,][\, W(u)Y'(u) - W'(u)Y(u)\,] \end{aligned} \tag{15}$$

satisfying $W(u) \neq 0$ on the interval I. (These roots never correspond to the degenerate case $W(u)X'(u) - W'(u)X(u) = W(u)Y'(u) - W'(u)Y(u) = 0$ with $W(u) \neq 0$ when the constraint (6) is imposed.)

Thus, in computing $\text{dist}(\mathbf{q}, \mathbf{r}(u))$ for a rational curve, we compare the values of the distance $|\,\mathbf{q} - \mathbf{r}(u)\,|$ at each odd root of (15) and at finite end points of the parameter interval I satisfying $W(u) \neq 0$ and/or at infinite end points in the case that $\deg(W) \geq \max(\deg(X), \deg(Y))$.

Remark 1.2 Note that equations (12) and (15) have an obvious geometric interpretation: in each case, the roots of the polynomial $P_{\perp}(u)$ identify points of the curve where lines drawn from $\mathbf{q}$ meet $\mathbf{r}(u)$ orthogonally. The distance (11) is then simply the smallest of the lengths of these perpendiculars (and the chords drawn from $\mathbf{q}$ to the affine end points of $\mathbf{r}(u)$, if any). Even-multiplicity roots of $P_{\perp}(u)$ can be ignored, since they identify points of $\mathbf{r}(u)$ where the distance $|\,\mathbf{q} - \mathbf{r}(u)\,|$ "levels off" but then continues to increase or decrease — *i.e.*, it does not attain a local extremum. Insisting that $\mathbf{r}(u)$ have a regular parameterization guarantees that $P_{\perp}(u)$ will not vanish in degenerate cases where $x'(u) = y'(u) = 0$ (which do not, in general, identify perpendiculars to $\mathbf{r}(u)$ from $\mathbf{q}$).

The preceding characterization of the point/curve distance function for polynomial and rational curves extends to any smooth analytic curve. In general, we write

$$\text{dist}(\mathbf{q}, \mathbf{r}(u)) \;=\; \min_{k \in \{1,\ldots,N\}} |\,\mathbf{q} - \mathbf{r}(u_k)\,|\,, \tag{16}$$

where $u_1, \ldots, u_N$ identify all the points on the analytic curve $\mathbf{r}(u)$ where a line drawn from $\mathbf{q}$ meets the curve orthogonally, as well as the affine end points (if any) of $\mathbf{r}(u)$. In the case of general analytic curves, of course, the determination of these parameter values will usually be more difficult than computing the odd roots of the polynomials (12) and (15) in the case of polynomial and rational curves.

Remark 1.3 For an algebraic curve C defined by the implicit polynomial equation $f(x, y) = 0$, we have

$$\text{dist}(\mathbf{q}, C) \;=\; \min_{k \in \{1,\ldots,N\}} |\,\mathbf{q} - \mathbf{r}_k\,|\,, \tag{17}$$

where $\mathbf{r}_1 = (x_1, y_1), \ldots, \mathbf{r}_N = (x_N, y_N)$ identify all points on the curve (*i.e.*, $f(x_k, y_k) = 0$) that satisfy

$$(a - x)\, f_y(x, y) \;-\; (b - y)\, f_x(x, y) \;=\; 0\,, \qquad (18)$$

f_x and f_y being the partial derivatives of f with respect to x and y. The points $\mathbf{r}_k$ are either singular points of $f(x, y) = 0$ (in the sense of an algebraic curve) or identify locations on $f(x, y) = 0$ where the line drawn from $\mathbf{q}$ meets the curve orthogonally (since the tangent at a point of $f(x, y) = 0$ is parallel to the vector $(f_y, -f_x)$ evaluated there [26, p. 55]).

Example 1.2 It is interesting to observe that, in operation, the procedure of Proposition 1.1 for computing $\mathrm{dist}(\mathbf{q}, \mathbf{r}(u))$ can run counter to geometric intuition. For example, one usually expects at the discrete parameter values $u_1, \ldots, u_N$ entering on the right–hand side of (13) that the curve $\mathbf{r}(u)$ *will lie locally to one side of its tangent line.* However, a simple counter–example to this notion is provided by the case $\mathbf{q} = (0, h)$ and $\mathbf{r}(u) = \{u, u^3\}$, for which $u = 0$ is always an odd–multiplicity root of $P_\perp(u)$ that should enter in (13), although the tangent line cuts the curve there. Moreover, in certain cases $\mathbf{r}(0)$ is actually the closest point of $\mathbf{r}(u)$ to $\mathbf{q}$. ■

We now note some important properties of the distance function:

Proposition 1.2 *When* $\mathbf{r}(u)$ *is a regular polynomial or rational curve, the function* $\mathrm{dist}(\mathbf{q}, \mathbf{r}(u))$ *is continuous — but not always differentiable — with respect to the location of the point* $\mathbf{q}$.

Proof : The continuity of $\mathrm{dist}(\mathbf{q}, \mathbf{r}(u))$ follows immediately from a general result concerning the distance between a point $\mathbf{q}$ and a non–empty set $\mathcal{S}$ in any metric space (see [16], Theorem 3, p. 53). However, it is instructive to examine this property in greater detail within the present context. For the sake of brevity we discuss only the case of a polynomial curve $\mathbf{r}(u)$ below; the extension to rational curves is relatively straightforward.

Consider expression (12) as a polynomial in *three* variables, namely, the parameter value u and the coordinates (a, b) of the point $\mathbf{q}$:

$$P_\perp(u, a, b) \;=\; [\, a - X(u)\,]\, X'(u) \;+\; [\, b - Y(u)\,]\, Y'(u)\,. \qquad (19)$$

At the reference point (a_0, b_0), let $u_{k,0} \in I$ be a simple[1] root of (19), so that $\partial P_\perp / \partial u \neq 0$ when $u = u_{k,0}$. Then by the *implicit function theorem* [2, p. 362] we infer the existence of a function $\phi_k(a, b)$, analytic in some two–dimensional neighborhood $\mathcal{N}_k$ of (a_0, b_0), such that $\phi_k(a_0, b_0) = u_{k,0}$ and

$$P_\perp(\phi_k(a,b), a, b) \equiv 0 \quad \text{for all } (a,b) \in \mathcal{N}_k \,. \tag{20}$$

Intuitively, the function $\phi_k(a,b)$ describes how the root u_k of $P_\perp(u)$ moves in the vicinity of its nominal value $u_{k,0}$ as the point $\mathbf{q} = (a,b)$ executes any path within the neighborhood $\mathcal{N}_k$ of its nominal location $\mathbf{q}_0 = (a_0, b_0)$.

(If u_k represents a finite end point of the parameter interval I on which $\mathbf{r}(u)$ is defined, rather than a simple root of (19), it can be incorporated into the above framework by simply taking $\phi_k(a,b) \equiv u_k$.)

Thus, about any nominal location $\mathbf{q}_0 = (a_0, b_0)$, we may invoke (13) to formulate the distance function in a neighborhood of that location as

$$\text{dist}(\mathbf{q}, \mathbf{r}(u)) \;=\; \min_{k \in \{1,\ldots,N\}} |\,\mathbf{q} - \mathbf{r}(\phi_k(\mathbf{q}))\,| \quad \text{for all } \mathbf{q} \in \mathcal{N} \,, \tag{21}$$

where $\mathcal{N} = \bigcap \mathcal{N}_k$ represents the area common to each of the neighborhoods of $\mathbf{q}_0 = (a_0, b_0)$ in which the root functions $\phi_k(\mathbf{q}) = \phi_k(a,b)$ are analytic.

In the formulation (21), the continuity of $\text{dist}(\mathbf{q}, \mathbf{r}(u))$ with respect to $\mathbf{q} = (a,b)$ at the (arbitrary) reference point $\mathbf{q}_0 = (a_0, b_0)$ is now apparent — each of the terms

$$|\,\mathbf{q} - \mathbf{r}(\phi_k(\mathbf{q}))\,| \;=\; \sqrt{[\,a - X(\phi_k(a,b))\,]^2 + [\,b - Y(\phi_k(a,b))\,]^2} \tag{22}$$

is continuous with respect to $\mathbf{q}$ at (a_0, b_0), since the functions $\phi_k(a,b)$ are analytic there and the curve $\mathbf{r}(u) = \{X(u), Y(u)\}$ is continuous everywhere, and although the index k that achieves the minimum in (21) may suddenly jump — from i to j, say — as we move through (a_0, b_0), we nevertheless have $|\,\mathbf{q} - \mathbf{r}(\phi_i(\mathbf{q}))\,| = |\,\mathbf{q} - \mathbf{r}(\phi_j(\mathbf{q}))\,|$ at any such jump.

If such a jump occurs in traversing (a_0, b_0), however, $\text{dist}(\mathbf{q}, \mathbf{r}(u))$ will not (in general) be differentiable with respect to $\mathbf{q}$ there. To see why, we consider the *directional derivative*

$$\mathbf{v} \cdot \nabla_\mathbf{q}\, \text{dist}(\mathbf{q}, \mathbf{r}(u)) \;=\; \left[\lambda \frac{\partial}{\partial a} + \mu \frac{\partial}{\partial b} \right] \text{dist}(\mathbf{q}, \mathbf{r}(u)) \,, \tag{23}$$

[1] For multiple roots the analysis is more involved; we shall not embark on it here.

which measures the rate of change of the distance function in the direction of the unit vector $\mathbf{v} = (\lambda, \mu)$ at the point (a_0, b_0) at which the partial derivatives in (23) are evaluated. Now by formal differentiation using the chain rule, and noting that $P_\perp(\phi_k(a,b), a, b) = 0$, the partial derivatives of the functions $\Delta_k(a,b) = |\,\mathbf{q} - \mathbf{r}(\phi_k(\mathbf{q}))\,|$ given by (22) may be expressed as

$$\frac{\partial \Delta_k}{\partial a} = \frac{a - X(\phi_k(a,b))}{\Delta_k(a,b)} \quad \text{and} \quad \frac{\partial \Delta_k}{\partial b} = \frac{b - Y(\phi_k(a,b))}{\Delta_k(a,b)}\,. \tag{24}$$

In (24) it is understood that k represents the index minimizing $\Delta_k(a,b)$, and if k jumps from i to j on passing through (a_0, b_0) in the direction $\mathbf{v}$, it is in general true that

$$X(\phi_i(a_0,b_0)) \neq X(\phi_j(a_0,b_0)) \quad \text{and} \quad Y(\phi_i(a_0,b_0)) \neq Y(\phi_j(a_0,b_0))\,, \tag{25}$$

although $\Delta_i(a_0,b_0) = \Delta_j(a_0,b_0)$. Therefore, the magnitude of the derivative (23) is discontinuous in general whenever we traverse a point (a_0, b_0) for which there is a jump in the index k that realizes the mimimum value on the right–hand side of expression (13). ∎

1.3 Point/curve bisectors

We are now ready to give a formal definition of point/curve bisectors:

Definition 1.3 *The bisector $B(\mathbf{p}, C)$ of a fixed point $\mathbf{p}$ and a plane curve C is the locus traced by a point that remains equidistant with respect to $\mathbf{p}$ and C, in the sense of the distance function* (11).

That the bisector of $\mathbf{p}$ and $\mathbf{r}(u)$ does indeed form a continuous locus can be seen from the preceding discussion. For if $\mathbf{q} = (a,b)$ is a point of the bisector for which $\text{dist}(\mathbf{q}, \mathbf{r}(u))$ is realized by a *unique* index k on the right–hand side of (13), then the direction $\mathbf{v} = (\lambda, \mu)$ of an infinitesimal displacement along the bisector — *i.e.*, the *tangent* to the bisector at $\mathbf{q}$ — is uniquely determined by the requirement of maintaining equal distance with respect to $\mathbf{p}$ and $\mathbf{r}(u)$, namely

$$\mathbf{v} \cdot \nabla_{\mathbf{q}}\, |\,\mathbf{q} - \mathbf{p}\,| \;=\; \mathbf{v} \cdot \nabla_{\mathbf{q}}\, \text{dist}(\mathbf{q}, \mathbf{r}(u))\,, \tag{26}$$

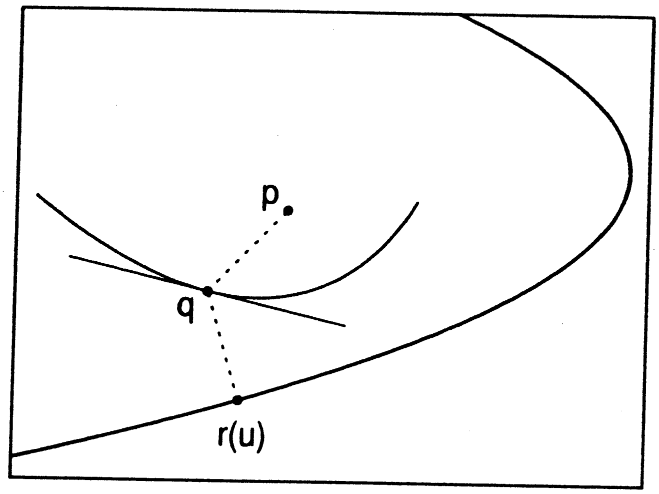

Figure 2: Local tangent direction for the bisector of $\mathbf{p}$ and $\mathbf{r}(u)$.

where $\mathbf{v}\cdot\nabla_{\mathbf{q}} = \lambda\partial/\partial a + \mu\partial/\partial b$ denotes the directional derivative with respect to $\mathbf{q} = (a, b)$ moving along $\mathbf{v} = (\lambda, \mu)$.

Now $\nabla_{\mathbf{q}} \,|\, \mathbf{q} - \mathbf{p} \,|$ is simply a unit vector in the direction of $\mathbf{q} - \mathbf{p}$, while $\nabla_{\mathbf{q}}\, \mathrm{dist}(\mathbf{q}, \mathbf{r}(u))$ is the unit normal $\mathbf{n}(u_k)$ to $\mathbf{r}(u)$ at the (unique) point u_k thereof for which $\mathrm{dist}(\mathbf{q}, \mathbf{r}(u))$ is realized. Thus, to satisfy (26), the direction of the tangent $\mathbf{v}$ to the bisector at $\mathbf{q}$ must be such as to *bisect the angle between the vectors* $\mathbf{q} - \mathbf{p}$ *and* $\mathbf{n}(u_k)$ (see Figure 2).

This provides a basis for numerically tracing the bisector, if the parameter value u_k at which $\mathrm{dist}(\mathbf{q}, \mathbf{r}(u)) = |\, \mathbf{q} - \mathbf{r}(u_k) \,|$ can be determined for any given $\mathbf{q}$. If this value is *not* unique, a more sophisticated analysis is required to choose among the possibilities (in general, the tangent is discontinuous at such points — the bisector is not smooth, although it is point–continuous). We shall not pursue this approach further, however.

2 Offset curves and bisectors

In formulating a tractable representation for the bisector of a point $\mathbf{p}$ and a curve $\mathbf{r}(u)$, it will be useful to recall the definition and some basic properties of the *offset curves* to $\mathbf{r}(u)$ (see [6, 7] for a more thorough discussion).

2.1 Constant–distance offsets

We begin by noting that if the curve $\mathbf{r}(u)$ is regular on the interval $u \in I$, its unit normal vector

$$\mathbf{n}(u) = \frac{(y'(u), -x'(u))}{\sqrt{x'^2(u) + y'^2(u)}} \tag{27}$$

is defined and continuous for all $u \in I$.

Definition 2.1 *The "untrimmed" offset at (signed) distance d to a regular parametric curve $\mathbf{r}(u)$ is the locus defined by*

$$\mathbf{r}_o(u) = \mathbf{r}(u) + d\,\mathbf{n}(u). \tag{28}$$

Note that when $\mathbf{r}(u)$ is a polynomial or rational curve, the offset $\mathbf{r}_o(u)$ is *not*, in general, a polynomial or rational curve, because of the radical in the denominator of (27). Consequently, offset curves are often approximated by piecewise–polynomial forms in computer–aided design (CAD) applications [15, 17, 21, 24], to render them compatible with existing representational and algorithmic infrastructures. (The "interior" and "exterior" offsets, at distances $-d$ and $+d$, together constitute an *algebraic curve* described by an implicit polynomial equation $f_o(x, y) = 0$ [7]. See also [8, 9] for discussion of a special class of polynomial curves whose offsets *are* rational.)

We call the locus (28) the "untrimmed" offset for the following reason: *Corresponding points* $\mathbf{r}(u)$ and $\mathbf{r}_o(u)$ on the given curve and its untrimmed offset are evidently distance d apart, measured along their mutual normal direction. However, the point $\mathbf{r}_o(u)$ of the untrimmed offset is not necessarily distance d, in the sense of the distance function (11), from the *entire curve* $\mathbf{r}(u)$. We shall call the locus having this latter property the "trimmed" offset to $\mathbf{r}(u)$, since it is obtained by deleting certain continuous segments of (28).

The trimming procedure may be characterized by the following property of the untrimmed offset curve (28):

Proposition 2.1 *For a regular polynomial or rational curve $\mathbf{r}(u)$ defined on the interval $u \in I$, let $\{i_1, \ldots, i_M\}$ be the ordered set of parameter values on I that correspond to self–intersections of its untrimmed offset $\mathbf{r}_o(u)$ at distance d, i.e., $\mathbf{r}_o(i_j) = \mathbf{r}_o(i_k)$ for some $1 \leq j \neq k \leq M$. Then, denoting the end points of I by i_0 and i_{M+1}, we have either*

$$\text{dist}(\mathbf{r}_o(t), \mathbf{r}(u)) \equiv d \quad \text{for all } t \in (i_k, i_{k+1}) \tag{29}$$

or

$$\text{dist}(\mathbf{r}_o(t), \mathbf{r}(u)) < d \quad \text{for all } t \in (i_k, i_{k+1}) \tag{30}$$

on each span (i_k, i_{k+1}) for $k = 0, \ldots, M$ between successive self–intersections of the untrimmed offset.

Proof : See Theorem 4.4 in [6]. ∎

Proposition 2.1 indicates that if we dissect the untrimmed offset $\mathbf{r}_o(u)$ into the subsegments delineated by its self–intersections, then each subsegment should be retained or discarded in its entirety in forming the trimmed offset. It is sufficient to test the distance of a single point interior to each span (i_k, i_{k+1}) of $\mathbf{r}_o(u)$ from the given curve $\mathbf{r}(u)$ (the mid point $\frac{1}{2}(i_k + i_{k+1})$, say) to determine whether or not that span should be eliminated (see Figure 3).

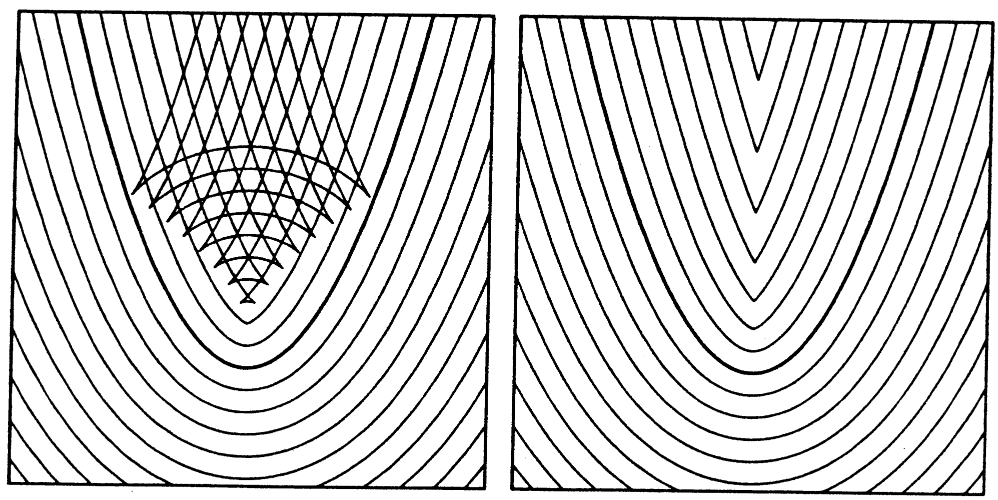

Figure 3: Untrimmed and trimmed offsets to a parabola.

Note that trimming an offset curve is a problem in the *global topology* of the locus defined by (28); we know of no simpler algorithm for the trimming process than the methodical dissect–and–test procedure described above. The parameter values $i_1, \ldots, i_M$ of the self–intersections can be specified as the roots of certain polynomials of rather high degree; see [7]. We shall encounter a similar "trimming" problem in computing point/curve bisectors.

Remark 2.1 The trimmed offsets at distance $\pm d$ to a given curve $\mathbf{r}(u)$ are the "level curves" for the point/curve distance function (11), *i.e.*, they are the loci of points $\mathbf{q}$ that satisfy $\operatorname{dist}(\mathbf{q}, \mathbf{r}(u)) = |d|$.

2.2 The "untrimmed" point/curve bisector

We can generalize the notion of an (untrimmed) offset curve at fixed distance d to a given regular curve $\mathbf{r}(u)$ by substituting any continuous function $d(u)$ of the parameter u in place of the constant d. The differentiability of the *variable–distance offset curve*

$$\mathbf{r}_o(u) = \mathbf{r}(u) + d(u)\mathbf{n}(u) \tag{31}$$

is then constrained by that of the "displacement function" $d(u)$. We shall find the form (31) to be valuable in analyzing point/curve bisectors.

Consider the *family of normal lines* to a given regular curve $\mathbf{r}(u)$. These lines may be parameterized in the form

$$\mathbf{r}(u) + \lambda\,\mathbf{n}(u)\,, \tag{32}$$

where u selects a point on the curve, and λ measures the signed distance along the normal line from that point. Given any point $\mathbf{p}$ not on $\mathbf{r}(u)$, the location $\mathbf{q}$ along (32) that is equidistant from $\mathbf{p}$ and the curve point $\mathbf{r}(u)$ is uniquely identified by the condition $|\,\mathbf{q} - \mathbf{r}(u)\,| = |\,\mathbf{q} - \mathbf{p}\,|$, which reduces to

$$|\lambda| = |\,\mathbf{r}(u) + \lambda\,\mathbf{n}(u) - \mathbf{p}\,|\,. \tag{33}$$

Now for each u, let $d(u)$ denote the unique value λ satisfying condition (33), and let $\psi(u)$ be the angle between the vector from $\mathbf{r}(u)$ to $\mathbf{p}$ and the normal $\mathbf{n}(u)$, measured in the right–handed sense defined by a unit vector $\mathbf{z}$ orthogonal to the plane of the curve. Referring to Figure 4, and noting that the points $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}(u)$ define an isosceles triangle, we have

$$d(u) = \tfrac{1}{2}\,|\,\mathbf{p} - \mathbf{r}(u)\,|\sec\psi(u) \tag{34}$$

by using the law of cosines. Since $(\mathbf{p} - \mathbf{r}(u)) \cdot \mathbf{n}(u) = |\,\mathbf{p} - \mathbf{r}(u)\,|\cos\psi(u)$, we can also express $d(u)$ as

$$d(u) = \frac{|\,\mathbf{p} - \mathbf{r}(u)\,|^2}{2\,(\mathbf{p} - \mathbf{r}(u)) \cdot \mathbf{n}(u)}\,. \tag{35}$$

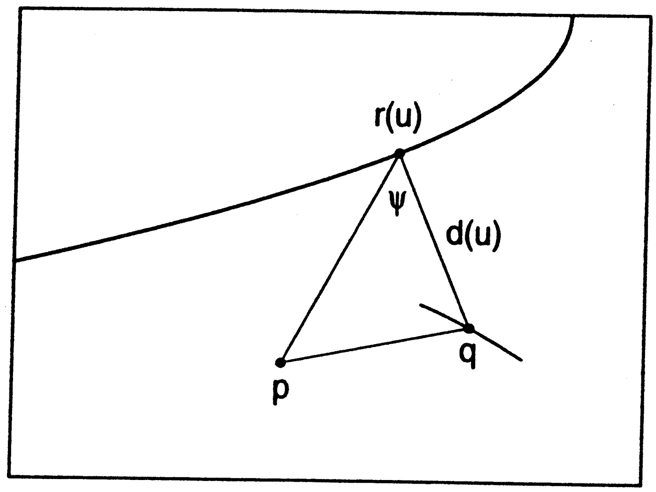

Figure 4: Definition of the displacement function $d(u)$.

If we regard the tangent line to the curve at $\mathbf{r}(u)$ as dividing the plane into two halves, it is evident from (35) that $d(u)$ will be positive or negative according to whether or not $\mathbf{p}$ lies in the half-plane that $\mathbf{n}(u)$ points in to.

Note that (35) is *not* (in general) a rational function of u, because of the radical incurred in computing the unit normal vector $\mathbf{n}(u)$.

Definition 2.2 *The untrimmed bisector of a fixed point* $\mathbf{p}$ *and a regular parametric curve* $\mathbf{r}(u)$ *is the variable-distance offset* (31) *to* $\mathbf{r}(u)$ *with the displacement function* (35).

Thus, the untrimmed bisector is simply the locus of points on the normal lines (32) that are equidistant from each curve point $\mathbf{r}(u)$ and the given point $\mathbf{p}$ (see Figure 4). We will show below that it is appropriately named, *i.e.*, it is a *superset* of the true bisector $B(\mathbf{p}, \mathbf{r}(u))$. Figure 5 illustrates the formulation of untrimmed point/curve bisectors as variable-distance offsets, in the simple case of a parabola.

Remark 2.2 When $\mathbf{r}(u)$ is a polynomial or rational curve, the untrimmed bisector defined by (31) and (35) has a *rational* parameterization, since the radicals in $d(u)$ and $\mathbf{n}(u)$ cancel each other.

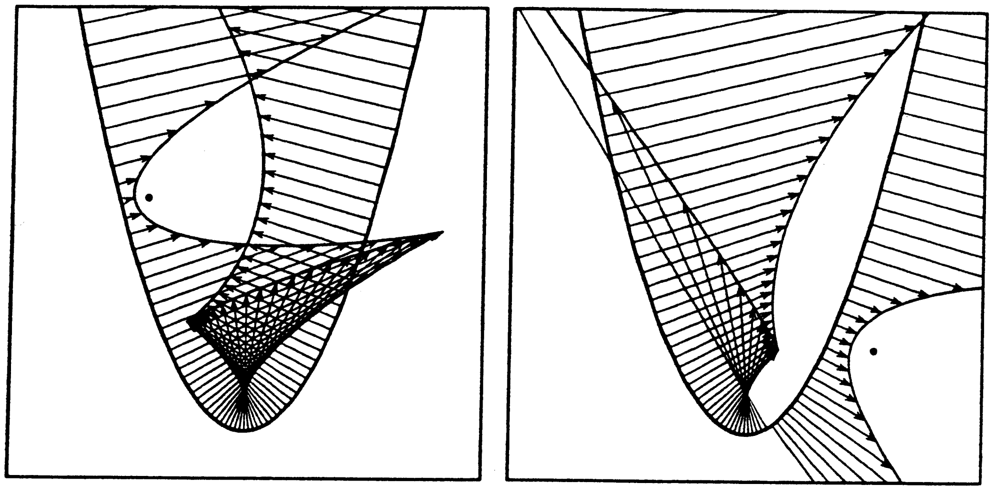

Figure 5: Untrimmed bisectors as variable–distance offsets.

Note that the displacement function satisfies $d(u) \neq 0$ for all u if the given point $\mathbf{p}$ does not lie on the curve $\mathbf{r}(u)$. However, the untrimmed bisector will exhibit a "point at infinity" for each parameter value τ that satisfies $(\mathbf{p} - \mathbf{r}(\tau)) \cdot \mathbf{n}(\tau) = 0$ (*i.e.*, the curve normal $\mathbf{n}(\tau)$ is orthogonal to the vector from $\mathbf{r}(\tau)$ to $\mathbf{p}$ or, equivalently, the tangent line at $\mathbf{r}(\tau)$ passes through $\mathbf{p}$). For the polynomial curve (3) satisfying (5), the parameter values corresponding to these points at infinity are the roots of the polynomial

$$P_\infty(u) \;=\; [\,\alpha - X(u)\,]\,Y'(u) \;-\; [\,\beta - Y(u)\,]\,X'(u)\,, \tag{36}$$

which is of degree $2n - 2$ (at most) when $\mathbf{r}(u)$ is of degree n. Similarly, for the rational curve (4), the polynomial whose roots identify points at infinity on the untrimmed bisector is

$$\begin{aligned} P_\infty(u) \;&=\; [\,\alpha W(u) - X(u)\,]\,[\,W(u)Y'(u) - W'(u)Y(u)\,] \\ &-\; [\,\beta W(u) - Y(u)\,]\,[\,W(u)X'(u) - W'(u)X(u)\,]\,. \end{aligned} \tag{37}$$

The roots of (37) identify only the *affine* points of $\mathbf{r}(u)$ that induce points at infinity on the untrimmed bisector — assuming that (6) holds. In addition, points at infinity of $\mathbf{r}(u)$, corresponding to the roots of $W(u)$, will give rise to points at infinity on the untrimmed bisector.

We will denote the untrimmed bisector defined by (31) and (35) by $\mathbf{b}(u)$, with homogeneous coordinates given by polynomials $X_b(u)$, $Y_b(u)$, $W_b(u)$. For the polynomial curve (3), it may be verified that

$$\begin{aligned} X_b &= [\,\alpha^2 - X^2 + (\beta - Y)^2\,]\,Y' - 2(\beta - Y)XX' , \\ Y_b &= 2(\alpha - X)YY' - [\,(\alpha - X)^2 + \beta^2 - Y^2\,]\,X' , \\ W_b &= 2\,[\,(\alpha - X)Y' - (\beta - Y)X'\,] , \end{aligned} \tag{38}$$

while in the case of the rational curve (4) we have

$$\begin{aligned} X_b &= [\,\alpha^2 W^2 - X^2 + (\beta W - Y)^2\,]\,V - 2(\beta W - Y)XU , \\ Y_b &= 2(\alpha W - X)YV - [\,(\alpha W - X)^2 + \beta^2 W^2 - Y^2\,]\,U , \\ W_b &= 2W[\,(\alpha W - X)V - (\beta W - Y)U\,] , \end{aligned} \tag{39}$$

where $U = WX' - W'X$ and $V = WY' - W'Y$. Note that $W_b \propto P_\infty$ in the case of a polynomial curve, while $W_b \propto WP_\infty$ in the rational case.

Remark 2.3 It may be verified from (38) and (39) that when $\mathbf{r}(u)$ is a *polynomial* curve of degree n, the untrimmed bisector $\mathbf{b}(u)$ is a rational curve of degree $3n - 1$ at most, whereas if $\mathbf{r}(u)$ is a *rational* curve of degree n, the untrimmed bisector is of degree $4n - 2$ at most.

Example 2.1 The simplest polynomial curve, other than a straight line, is the parabola. Consider the case $\mathbf{p} = (\alpha, \beta)$ and $\mathbf{r}(u) = \{u, u^2\}$. From equations (38), we have the representation

$$\begin{aligned} X_b(u) &= 2u\,(u^4 - 2\beta u^2 + \alpha^2 + \beta^2 - \beta) , \\ Y_b(u) &= -\,3u^4 + 4\alpha u^3 - u^2 + 2\alpha u - \alpha^2 - \beta^2 , \\ W_b(u) &= -\,2\,(u^2 - 2\alpha u + \beta) , \end{aligned} \tag{40}$$

for the untrimmed bisector, which is evidently a rational curve of degree *five*. Note that the roots of the denominator polynomial $W_b(u)$ are simply

$$u = \alpha \pm \sqrt{\alpha^2 - \beta} \tag{41}$$

which identify real points at infinity of (40) at *finite* parameter values when $\beta < \alpha^2$, *i.e.*, $\mathbf{p}$ lies "outside" the parabola. Since $\max(\deg(X_b), \deg(Y_b)) > \deg(W_b)$, the parameter values $u = \pm\infty$ also identify points at infinity on the untrimmed bisector (40) for any (α, β). Figure 6 illustrates representative examples of the curves defined by (40). ∎

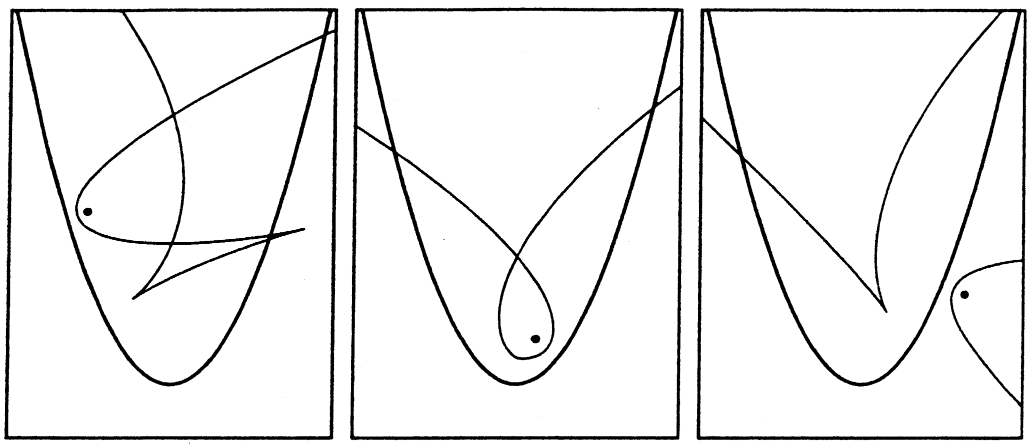

Figure 6: Untrimmed bisectors of a point and a parabola.

Example 2.2 As a simple example of the untrimmed bisector of a point and a rational curve, we consider the case of an ellipse centered on the origin, with semi–axes 1 and k. This has the rational parameterization

$$X(u) = 1 - u^2, \quad Y(u) = 2k\,u, \quad W(u) = 1 + u^2. \tag{42}$$

Substituting the above into (39), we find that the untrimmed bisector is a rational curve of degree *six*, defined by

$$\begin{aligned} X_b(u) &= (1-u^2)\,[\,k(\alpha^2+\beta^2-1)u^4 + 4(1-k^2)\beta u^3 \\ &\quad + 2k(\alpha^2+\beta^2-3+2k^2)u^2 + 4(1-k^2)\beta u + k(\alpha^2+\beta^2-1)\,], \\ Y_b(u) &= 2u\,[\,(\alpha^2+\beta^2+2(1-k^2)\alpha+1-2k^2)u^4 \\ &\quad + 2(\alpha^2+\beta^2-1)u^2 + \alpha^2+\beta^2-2(1-k^2)\alpha+1-2k^2\,], \\ W_b(u) &= 2(1+u^2)^2\,[-k(\alpha+1)u^2 + 2\beta u + k(\alpha-1)\,]. \end{aligned} \tag{43}$$

In this case, the untrimmed bisector will have real points at infinity only when the condition $\alpha^2 + (\beta/k)^2 > 1$ is satisfied, *i.e.*, $\mathbf{p}$ lies *outside* the ellipse. They occur at the parameter values

$$u = \frac{\beta \pm \sqrt{k^2\alpha^2 + \beta^2 - k^2}}{k(\alpha+1)}. \tag{44}$$

Some examples are illustrated in Figure 7. ∎

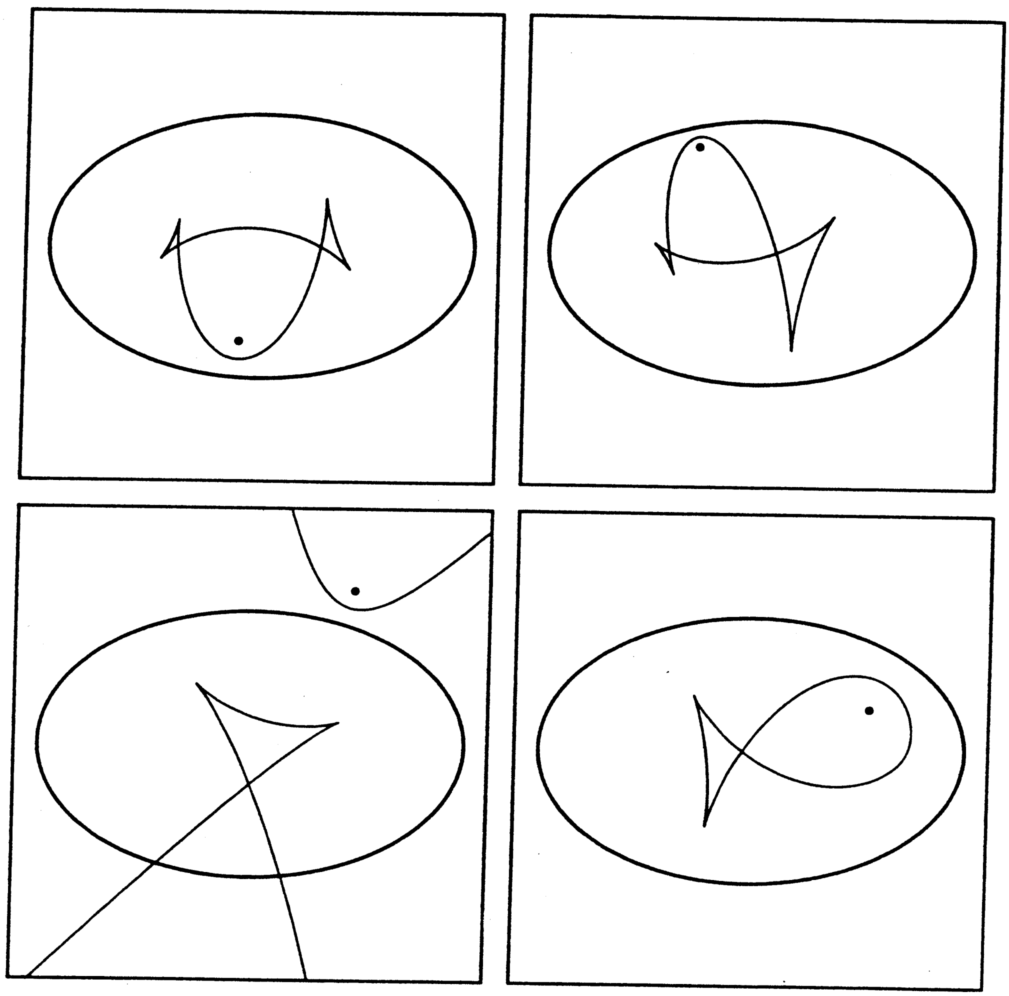

Figure 7: Untrimmed bisectors of a point and an ellipse.

2.3 Irregular points of the untrimmed bisector

It is evident from Figures 6 and 7 that, in general, the untrimmed bisector of a point $\mathbf{p}$ and a regular curve $\mathbf{r}(u)$ is *not* a smooth locus, even though $\mathbf{r}(u)$

is necessarily smooth if it is regular. Apart from their intrinsic interest, an understanding of the irregular points or "cusps" of the untrimmed bisector is important for the trimming procedure (see Section 3 below).

Recall [18] that for any regular parametric curve $\mathbf{r}(u)$, the elementary differential characteristics at each point may be expressed in terms of the parametric derivatives $\mathbf{r}'(u), \mathbf{r}''(u), \ldots$ there as

$$\mathbf{t} = \frac{\mathbf{r}'}{|\mathbf{r}'|}, \quad \mathbf{n} = \mathbf{t}\times\mathbf{z}, \quad \kappa = \frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{z}}{|\mathbf{r}'|^3}, \tag{45}$$

$\mathbf{z}$ being a unit vector orthogonal to the plane of the curve. The *normal* $\mathbf{n}(u)$ and *tangent* $\mathbf{t}(u)$ form an orthonormal basis $(\mathbf{n},\mathbf{t},\mathbf{z})$ with $\mathbf{z}$ at each point u, while $\kappa(u)$ is the (signed) *curvature*. The variation of the tangent and normal along the curve is described by the *Frenet equations*:

$$\mathbf{t}' = -\sigma\kappa\,\mathbf{n} \quad \text{and} \quad \mathbf{n}' = \sigma\kappa\,\mathbf{t}, \tag{46}$$

where σ is the parametric speed (2) of $\mathbf{r}(u)$. Higher–order derivatives of $\mathbf{t}(u)$ and $\mathbf{n}(u)$ are readily expressed in terms of $\mathbf{t}(u)$ and $\mathbf{n}(u)$ and the scalar functions $\sigma(u)$, $\kappa(u)$, and their derivatives — for example,

$$\begin{aligned}\mathbf{t}'' &= -\sigma^2\kappa^2\,\mathbf{t} - (\sigma'\kappa+\sigma\kappa')\,\mathbf{n}, \\ \mathbf{n}'' &= (\sigma'\kappa+\sigma\kappa')\,\mathbf{t} - \sigma^2\kappa^2\,\mathbf{n}.\end{aligned} \tag{47}$$

If we denote by $\mathbf{b}(u)$ the parametric representation of the untrimmed bisector obtained by substituting from (35) into (31), then the parametric derivatives of $\mathbf{b}(u)$ may be written as

$$\begin{aligned}\mathbf{b}' &= \mathbf{r}' + d'\mathbf{n} + d\,\mathbf{n}', \\ \mathbf{b}'' &= \mathbf{r}'' + d''\mathbf{n} + 2d'\mathbf{n}' + d\,\mathbf{n}'',\end{aligned} \tag{48}$$

... etc. Setting $\mathbf{r}' = \sigma\mathbf{t}$ and $\mathbf{r}'' = \sigma'\mathbf{t} - \sigma^2\kappa\mathbf{n}$ and substituting from (46) and (47), we can re–write (48) as

$$\begin{aligned}\mathbf{b}' &= \sigma(1+\kappa d)\,\mathbf{t} + d'\,\mathbf{n}, \\ \mathbf{b}'' &= [\,\sigma'(1+\kappa d) + \sigma(\kappa' d + 2\kappa d')\,]\,\mathbf{t} \\ &\quad + [\,d'' - \sigma^2\kappa(1+\kappa d)\,]\,\mathbf{n},\end{aligned} \tag{49}$$

where, using the form (34), the derivatives of the displacement function $d(u)$ appropriate to the untrimmed bisector are most conveniently expressed as

$$\begin{aligned} d' &= \sigma(1+\kappa d)\tan\psi\,, \\ d'' &= [\,\sigma'(1+\kappa d)+\sigma\kappa' d\,]\tan\psi \\ &\quad + \frac{\sigma^2(1+\kappa d)}{2d}[\,1+2\kappa d+(1+4\kappa d)\tan^2\psi\,]\,, \end{aligned} \tag{50}$$

... etc. In deriving (50), we make use of the fact that the angle ψ between the vector from $\mathbf{r}(u)$ to $\mathbf{p}$ and the curve normal $\mathbf{n}(u)$ changes at the rate

$$\psi' = \frac{\sigma(1+2\kappa d)}{2d} \tag{51}$$

with respect to u, as may be deduced by differentiating the relation $\cos\psi = (\mathbf{p}-\mathbf{r}(u))\cdot\mathbf{n}(u)\,/\,|\,\mathbf{p}-\mathbf{r}(u)\,|$.

Using the above expression for d', we now see that the first parametric derivative of the untrimmed bisector has the form

$$\mathbf{b}' = \sigma(1+\kappa d)\,(\mathbf{t}+\mathbf{n}\tan\psi)\,. \tag{52}$$

At each value of u such that $\mathbf{b}'(u)\neq\mathbf{0}$, the tangent $\mathbf{t}_b(u)$ to the untrimmed bisector is a unit vector in the direction of (52). Note that the magnitude of (52), *i.e.*, the *parametric speed* $\sigma_b(u)$ of the untrimmed bisector, is simply

$$|\mathbf{b}'| = \sigma\,|\,1+\kappa d\,|\,|\sec\psi\,|\,. \tag{53}$$

Since the given curve $\mathbf{r}(u)$ is regular, its tangent $\mathbf{t}(u)$ and normal $\mathbf{n}(u)$ are defined and linearly independent at each u, so $\mathbf{t}(u)+\mathbf{n}(u)\tan\psi(u)$ is never the zero vector. Moreover, this vector varies continuously with u except at those parameter values where $\psi(u)=\pm\pi/2$, which correspond to the roots of the polynomial (36), *i.e.*, the points at infinity on $\mathbf{b}(u)$. Hence, at each u such that $P_\infty(u)\neq 0$, the unit vector

$$\mathbf{v}(u) = |\cos\psi(u)\,|\,[\,\mathbf{t}(u)+\mathbf{n}(u)\tan\psi(u)\,] \tag{54}$$

in the direction of $\mathbf{t}(u)+\mathbf{n}(u)\tan\psi(u)$ is defined and varies continuously with u. Since $\sigma(u)\neq 0$ for all u on a regular curve, we see that the tangent $\mathbf{t}_b(u)=\mathbf{b}'(u)/|\mathbf{b}'(u)|$ to the untrimmed bisector is given in terms of $\mathbf{v}(u)$ by

$$\mathbf{t}_b(u) = \frac{1+\kappa(u)d(u)}{|\,1+\kappa(u)d(u)\,|}\,\mathbf{v}(u)\,. \tag{55}$$

Lemma 2.1 *The untrimmed bisector* $\mathbf{b}(u)$ *exhibits a* cusp, *or sudden tangent reversal, at those parameter values where* $P_\infty(u) \neq 0$ *and the curvature* $\kappa(u)$ *of the given curve* $\mathbf{r}(u)$ *attains the local critical value*

$$\kappa_c(u) = -\frac{1}{d(u)} = -\frac{2\cos\psi(u)}{|\,\mathbf{p}-\mathbf{r}(u)\,|}, \tag{56}$$

without being an extremum, i.e., $\kappa'(u) \neq 0$.

Proof : Let τ be a parameter value such that $P_\infty(\tau) \neq 0$ (*i.e.*, $\mathbf{b}(\tau)$ is an affine point of the untrimmed bisector), and the curvature of the given curve $\mathbf{r}(u)$ satisfies both $\kappa(\tau) = -1/d(\tau)$ and $\kappa'(\tau) \neq 0$. Then $\tan\psi(\tau)$ is finite, and the unit vector $\mathbf{v}(\tau)$ given by (54) is defined and continuous at $u = \tau$. On the other hand, the scalar factor multiplying $\mathbf{v}(u)$ in (55) is a "step function," which changes abruptly from -1 to $+1$, or vice-versa, at $u = \tau$ whenever $\mathrm{d}/\mathrm{d}u[\,1+\kappa(u)d(u)\,] \neq 0$ at $u = \tau$ (*i.e.*, $\kappa'(\tau)d(\tau) + \kappa(\tau)d'(\tau) \neq 0$). But from (50) we observe that $d'(\tau) = 0$ whenever $\kappa(\tau) = -1/d(\tau)$, and since we certainly have $d(\tau) \neq 0$ because $\mathbf{p}$ does not lie on $\mathbf{r}(u)$, the condition $\mathrm{d}/\mathrm{d}u[\,1+\kappa(u)d(u)\,] \neq 0$ at $u = \tau$ is exactly equivalent to $\kappa'(\tau) \neq 0$. ∎

Remark 2.4 It is interesting to note that the criteria $\kappa(u) = -1/d(u)$ and $\kappa'(u) \neq 0$ identifying the cusps of the "variable–distance" offset (31) are identical to those for fixed–distance offsets ($\kappa(u) = -1/d$, $\kappa'(u) \neq 0$) with $d(u) =$ constant [6]. This is not a *generic* feature of variable offsets, but arises rather from the specific form (35) of $d(u)$ for the untrimmed bisector.

Using (35) and (45), we see that the local critical curvature (56) will be attained at those roots of the equation

$$\Big[\,|\,\mathbf{p}-\mathbf{r}(u)\,|^2\,\mathbf{r}'(u)\times\mathbf{r}''(u) + 2\,|\mathbf{r}'(u)|^2\,[\,\mathbf{p}-\mathbf{r}(u)\,]\times\mathbf{r}'(u)\,\Big]\cdot\mathbf{z} = 0 \tag{57}$$

that satisfy $\kappa'(u) \neq 0$. These parameter values correspond to cusps on the untrimmed bisector $\mathbf{b}(u)$. When $\mathbf{r}(u)$ is the polynomial curve (3), equation (57) corresponds to a polynomial of degree $4n-4$ (at most) in u:

$$\begin{aligned} P_c = {} & [\,(\alpha-X)^2+(\beta-Y)^2\,]\,(X'Y''-X''Y') \\ & + 2\,(X'^2+Y'^2)\,[\,(\alpha-X)Y'-(\beta-Y)X'\,]\,. \end{aligned} \tag{58}$$

If $\mathbf{r}(u)$ is the rational curve (4), the left–hand side of (57) is a rational function in u, whose numerator is a polynomial of degree $7n-6$ (at most) in u:

$$\begin{aligned} P_c \;=\; & W\,[\,(\alpha W - X)^2 + (\beta W - Y)^2\,]\,(U_1V_2 - U_2V_1) \\ & + 2\,(U_1^2 + V_1^2)\,[\,(\alpha W - X)V_1 - (\beta W - Y)U_1\,]\,, \end{aligned} \qquad (59)$$

where we denote $(WX'-W'X, WY'-W'Y)$ and $(WX''-W''X, WY''-W''Y)$ by (U_1, V_1) and (U_2, V_2), respectively, for brevity.

Consider now the behavior of the curvature $\kappa_b(u)$ along the untrimmed bisector. By substituting from (49) and (50), a straightforward but laborious calculation gives

$$(\mathbf{b}' \times \mathbf{b}'') \cdot \mathbf{z} \;=\; -\,\frac{\sigma^3(1+\kappa d\,)^2}{2d}\,\sec^2\psi\,, \qquad (60)$$

and together with (53) the expression $\kappa_b(u) = |\mathbf{b}'(u)|^{-3}[\,\mathbf{b}'(u) \times \mathbf{b}''(u)\,] \cdot \mathbf{z}$ for the curvature of the untrimmed bisector reduces to

$$\kappa_b \;=\; -\,\frac{|\cos\psi\,|}{2d\,|\,1+\kappa d\,|}\,. \qquad (61)$$

It is worthwhile emphasizing the significance of equation (61) in words: at any point u on the untrimmed bisector, we can express the curvature $\kappa_b(u)$ of $\mathbf{b}(u)$ simply in terms of the curvature $\kappa(u)$ of the given curve $\mathbf{r}(u)$, the displacement function $d(u)$ defined by (35), and the angle $\psi(u)$ between the vector $\mathbf{p} - \mathbf{r}(u)$ and the curve normal $\mathbf{n}(u)$.

Lemma 2.2 *The untrimmed bisector* $\mathbf{b}(u)$ *has an* extraordinary point — *i.e., a tangent-continuous point of infinite curvature — at those parameter values* u *where* $P_\infty(u) \neq 0$ *and the curvature* $\kappa(u)$ *of the given curve* $\mathbf{r}(u)$ *attains an extremum* ($\kappa'(u) = 0$ *but* $\kappa''(u) \neq 0$) *equal in value to the local critical curvature* $\kappa_c(u)$ *defined by* (56).

Proof : Let τ be such that $P_\infty(\tau) \neq 0$ and the curvature of $\mathbf{r}(u)$ satisfies $\kappa(\tau) = -1/d(\tau)$ with $\kappa'(\tau) = 0 \neq \kappa''(\tau)$. From equations (50) it is clear that

$$1 + \kappa(\tau)d(\tau) \;=\; \kappa'(\tau) \;=\; 0 \;\Rightarrow\; d'(\tau) \;=\; d''(\tau) \;=\; 0\,. \qquad (62)$$

Thus, the first derivative $\kappa'(\tau)d(\tau) + \kappa(\tau)d'(\tau)$ of the quantity $1+\kappa d$ at $u = \tau$ is zero, whereas the second derivative, $\kappa''(\tau)d(\tau) + 2\kappa'(\tau)d'(\tau) + \kappa(\tau)d''(\tau)$, is non–vanishing (since $\kappa''(\tau) \neq 0$, and $d(u) \neq 0$ for all u). Hence the factor $(1+\kappa d)/|1+\kappa d|$ multiplying the unit vector $\mathbf{v}(\tau)$ in expression (55) has the *same* sign on either side of τ, and the untrimmed bisector tangent $\mathbf{t}_b(u)$ is therefore *continuous* at $u = \tau$. However, since $1+\kappa(\tau)d(\tau) = 0$, it is evident from (61) that the curvature $\kappa_b(u)$ of the untrimmed bisector will increase without bound as we approach τ. (Note in (61) that $\cos\psi \neq 0$, since τ is not a root of $P_\infty(u)$, and d is never zero if the point $\mathbf{p}$ does not lie on $\mathbf{r}(u)$.) ∎

We may regard extraordinary points, and all other irregular points of the untrimmed bisector generated by points of $\mathbf{r}(u)$ that satisfy

$$\kappa(u) = -1/d(u) \quad \text{and} \quad \kappa'(u) = \cdots = \kappa^{(r)}(u) = 0 \neq \kappa^{(r+1)}(u) \tag{63}$$

with $r > 1$, as "higher–order" cusps. Unlike the simple or "ordinary" cusps, the occurrence of such points is exceptional.

Example 2.3 In the case of the parabola $\mathbf{r}(u) = \{u, u^2\}$ of Example 2.1, the polynomial (58) is the quartic

$$P_c(u) \;=\; 3u^4 - 8\alpha u^3 + 6\beta u^2 - (\alpha^2 + \beta^2 - \beta)\,. \tag{64}$$

Although $P_c(u)$ is of degree four, *Descartes Law of Signs* [25] and a careful analysis of the behavior of its coefficients for real values of α and β reveals that it has just two distinct real roots when $\alpha^2+\beta^2-\beta > 0$, and none when $\alpha^2+\beta^2-\beta < 0$. Exceptionally, if $\alpha^2+\beta^2-\beta = 0$, $P_c(u)$ has a double root at $u = 0$, corresponding to an extraordinary point of the untrimmed bisector; this occurs when (α,β) *lies on the circle of curvature to the vertex* $(0,0)$ of the parabola.

For the ellipse (42) in Example 2.2, the "cusp polynomial" becomes

$$\begin{aligned} P_c(u) \;=\;& k\,[\,\alpha^2+\beta^2+2(1-k^2)\alpha+1-2k^2\,]\,u^6 \\ +\;& 3k\,[\,\alpha^2+\beta^2-2(1-k^2)\alpha-3+2k^2\,]\,u^4 \\ +\;& 16\,(1-k^2)\beta\,u^3 \\ +\;& 3k\,[\,\alpha^2+\beta^2+2(1-k^2)\alpha-3+2k^2\,]\,u^2 \\ +\;& k\,[\,\alpha^2+\beta^2-2(1-k^2)\alpha+1-2k^2\,]\,. \end{aligned} \tag{65}$$

In this case the maximum number of distinct real roots is four, again less than is suggested by the degree of $P_c(u)$. Note that we have double roots at zero or infinity, giving rise to extraordinary points, when (α, β) is such that the coefficient of u^0 or u^6 vanishes, respectively. (The coefficients of these terms define the circles of curvature to the ellipse at the vertices $u = 0$ and $u = \pm\infty$; see the discussion in Example 3.2 below.) ∎

2.4 The true bisector

We now show that the untrimmed bisector given by (31) and (35) is a superset of the "true" bisector (Definition 1.3). Note that the "true" bisector of $\mathbf{p}$ and $\mathbf{r}(u)$ can be visualized as follows. Consider a circle through $\mathbf{p}$, increasing in size until it just touches the curve $\mathbf{r}(u)$. Now consider all such circles, inflating in all directions from $\mathbf{p}$ until they touch $\mathbf{r}(u)$. The bisector is the locus of the centers of these maximal circles. In the following discussion, it is useful to keep this representation of the bisector in mind.

Definition 2.3 *Let $C_{\mathbf{q}}$ denote the circle with center $\mathbf{q}$ and radius $|\,\mathbf{q} - \mathbf{p}\,|$.*

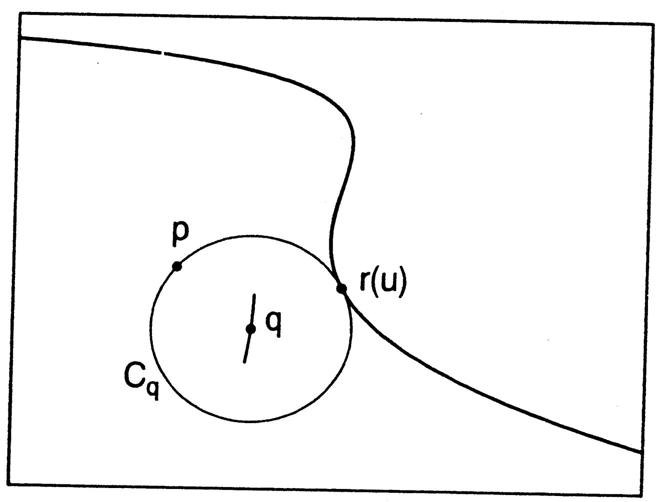

Figure 8: Condition for a point $\mathbf{q}$ to lie on the bisector $B(\mathbf{p}, \mathbf{r}(u))$.

Remark 2.5 A point $\mathbf{q}$ lies on the bisector of $\mathbf{p}$ and the regular curve $\mathbf{r}(u)$ if and only if (see Figure 8):

- $C_{\mathbf{q}}$ is "empty" — no point of $\mathbf{r}(u)$ lies in its interior; and
- $C_{\mathbf{q}}$ is tangent to $\mathbf{r}(u)$ in at least one point.

Proof : $\mathbf{q}$ lies on the bisector if and only if the closest point of $\mathbf{r}(u)$ to $\mathbf{q}$ is at distance $|\mathbf{q}-\mathbf{p}|$. If $C_{\mathbf{q}}$ is tangent to $\mathbf{r}(u)$ at $u = u_0$ but otherwise empty, then $\mathbf{r}(u_0)$ is closest to $\mathbf{q}$ and is at distance $|\mathbf{q} - \mathbf{r}(u_0)| = |\mathbf{q}-\mathbf{p}|$. ∎

(Note that if we are computing the bisector of a point and a finite curve segment, the circle $C_{\mathbf{q}}$ may contain curve points that correspond to parameter values outside the domain I that defines the segment of interest.)

Proposition 2.2 *The untrimmed bisector of* $\mathbf{p}$ *and* $\mathbf{r}(u)$ *is a superset of the true bisector of* $\mathbf{p}$ *and* $\mathbf{r}(u)$.

Proof : Let $\mathbf{q}$ be a point of the bisector of $\mathbf{p}$ and $\mathbf{r}(u)$. By Remark 2.5, there exists a point $\mathbf{r}(u_0)$ of the curve that lies on the circle $C_{\mathbf{q}}$, such that the curve is tangent to the circle at $\mathbf{r}(u_0)$ or, equivalently, such that the normal at $\mathbf{r}(u_0)$ passes through the center $\mathbf{q}$ of the circle $C_{\mathbf{q}}$. Thus, $\mathbf{r}(u_0) + d(u_0)\mathbf{n}(u_0) = \mathbf{q}$ (recall that $d(u_0) = |\mathbf{q}-\mathbf{p}|$ = radius of $C_{\mathbf{q}}$). ∎

Since the second condition of Remark 2.5 is satisfied for *all* points $\mathbf{q}$ of the untrimmed bisector (by definition), a point $\mathbf{q}$ of the untrimmed bisector is a point of the true bisector if and only if the interior of $C_{\mathbf{q}}$ is empty.

3 The trimming procedure

The untrimmed bisector $\mathbf{b}(u)$ may be "trimmed" down to the true bisector by deleting a finite number of segments. As with the untrimmed offset, this is done by finding a finite number of special points that identify possible deviations of $\mathbf{b}(u)$ from the true bisector. There are four classes of these special points on the untrimmed bisector (see Definitions 3.3 and 3.4), and we split the trimming process into two stages. We now describe the first stage, which removes "inactive" segments.

3.1 Active and inactive segments

Along the normal line to every curve point $\mathbf{r}(u_0)$, there is a corresponding point $\mathbf{b}(u_0)$ of the untrimmed bisector. Many of these points $\mathbf{b}(u_0)$ do not belong to the true bisector, however, because $\mathbf{r}(u_0)$ is clearly not the closest point of the curve to $\mathbf{b}(u_0)$ (see Figure 9).

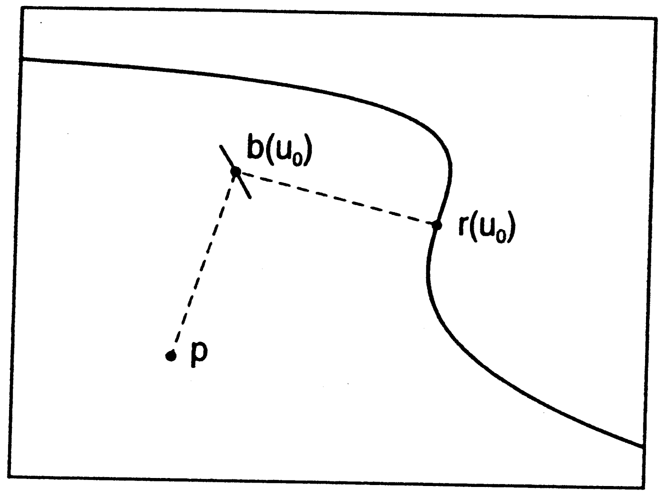

Figure 9: $\mathbf{r}(u_0)$ is *not* closest on $\mathbf{r}(u)$ to its corresponding point $\mathbf{b}(u_0)$.

Definition 3.1 *The points* $\mathbf{r}(u_0)$ *and* $\mathbf{b}(u_0) = \mathbf{r}(u_0) + d(u_0)\mathbf{n}(u_0)$ *are called* corresponding points *of the given curve and the untrimmed bisector.*

Definition 3.2 *The point* $\mathbf{q} = \mathbf{b}(u_0)$ *of the untrimmed bisector is* active *if either of the following conditions holds:*

(1) $\mathbf{q}$ *has more than one corresponding point on the curve*[2]

(2) $\mathbf{q}$ *has only one corresponding point* $\mathbf{q}' = \mathbf{r}(u_0)$ *on the curve, and either*

- (a) *the circle of curvature at* $\mathbf{q}'$ *contains the point* $\mathbf{p}$ *— i.e.,* $\mathbf{p}$ *lies on or inside the circle of curvature, or*
- (b) *the point* $\mathbf{p}$ *and the circle of curvature at* $\mathbf{q}'$ *lie on opposite sides of the tangent at* $\mathbf{q}'$

(see Figure 10). A segment S *of* $\mathbf{b}(u)$ *is active if every point of* S *is active.*

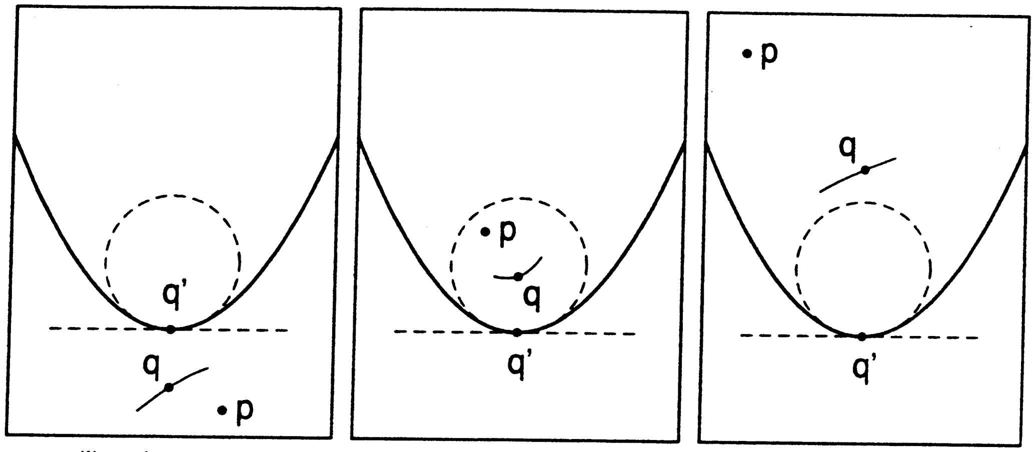

Figure 10: Active and inactive points on the untrimmed bisector.

To understand the significance of such "active" points on the untrimmed bisector, we need the following result:

Lemma 3.1 *Let* $\mathbf{q}$ *and* $\mathbf{q}'$ *be corresponding points on the untrimmed bisector and the given curve. Then the circle* $C_{\mathbf{q}}$ *lies entirely inside/outside the circle of curvature at* $\mathbf{q}'$ *according to whether the point* $\mathbf{p}$ *lies inside/outside this circle of curvature.*

Proof : Let $\mathcal{C}$ denote the circle of curvature to $\mathbf{r}(u)$ at the point $\mathbf{q}'$. Then the normal line to $\mathbf{r}(u)$ at $\mathbf{q}'$ is a diameter of $\mathcal{C}$, and the point $\mathbf{q}$ corresponding to $\mathbf{q}'$ lies on this normal line. Noting that $\mathbf{q}$ is the center of the circle $C_{\mathbf{q}}$ and that (by construction) both $\mathbf{q}'$ and $\mathbf{p}$ lie on this circle, we consider a variable circle constrained to pass through $\mathbf{q}'$ whose center moves away from $\mathbf{q}'$ along the normal line there. Such a circle evidently lies entirely inside or outside of (or, exceptionally, coincides with) the circle of curvature $\mathcal{C}$ at $\mathbf{q}'$. $C_{\mathbf{q}}$ is an instance of such a circle — therefore, the inclusion/exclusion of any point of $C_{\mathbf{q}}$ in $\mathcal{C}$ is sufficient to guarantee that all of $C_{\mathbf{q}}$ lies inside/outside $\mathcal{C}$. As already noted, $\mathbf{p}$ is a point of $C_{\mathbf{q}}$. ∎

[2]There are a finite number of such points; we will have more to say about them below.

Remark 3.1 We observe that an active point appears to lie on the bisector, at least locally. To see this, let $\mathbf{q}$ be an active point of $\mathbf{b}(u)$ with only one corresponding point $\mathbf{q}'$. If $\mathbf{p}$ lies inside or on the circle of curvature at $\mathbf{q}'$, then by Lemma 3.1 all of $C_\mathbf{q}$ also lies in the circle of curvature at $\mathbf{q}'$ and, in particular, there exists a neighborhood of $\mathbf{q}'$ that lies completely outside of $C_\mathbf{q}$ [12, p. 176]. On the other hand, if $\mathbf{p}$ (and thus $C_\mathbf{q}$) lies on the opposite side of the tangent at $\mathbf{q}'$ from the circle of curvature at $\mathbf{q}'$, then again the curve in a neighborhood of $\mathbf{q}'$ lies completely outside of $C_\mathbf{q}$. Thus, the curve in some neighborhood of an active point (with one corresponding point) lies completely outside of $C_\mathbf{q}$. In other words, an active point acts at least locally like a point of the bisector (Remark 2.5).

Proposition 3.1 *An inactive point of the untrimmed bisector of* $\mathbf{p}$ *and* $\mathbf{r}(u)$ *does not lie on the bisector of* $\mathbf{p}$ *and* $\mathbf{r}(u)$.

Proof : Let $\mathbf{q}$ be an inactive point of $\mathbf{b}(u)$ and let $\mathbf{q}'$ be the point of $\mathbf{r}(u)$ corresponding to $\mathbf{q}$. By definition, $\mathbf{p}$ and the circle of curvature at $\mathbf{q}'$ lie on the same side of the tangent at $\mathbf{q}'$, but $\mathbf{p}$ lies outside this circle (Figure 10(iii)). $C_\mathbf{q}$ and the circle of curvature at $\mathbf{q}'$ are both circles that are tangent to $\mathbf{q}'$ (or, equivalently, circles through $\mathbf{q}'$ with centers on the normal line there), and they both lie on the same side of the tangent at $\mathbf{q}'$. Moreover, by Lemma 3.1, $C_\mathbf{q}$ must be strictly larger than (and strictly contain) the circle of curvature at $\mathbf{q}'$. But all circles tangent at $\mathbf{q}'$ that are larger than the circle of curvature there must lie on one side of the curve in some neighborhood of $\mathbf{q}'$ [12, p. 176], and consequently $C_\mathbf{q}$ must contain some points of the curve in its interior. We conclude that $\mathbf{q}$ is not on the bisector (Remark 2.5). ∎

Since the definition of an active point depends on the side of the tangent (at $\mathbf{q}'$) that $\mathbf{p}$ and the circle of curvature lie on, we are interested in points where this can change. We are also interested in points where the position of $\mathbf{p}$ relative to the circle of curvature can change.

Definition 3.3 *A point of the curve* $\mathbf{r}(u)$ *is an* inflection *if the curvature is zero at that point; a point of* $\mathbf{r}(u)$ *is a* class point[3] *if the tangent at that point passes through* $\mathbf{p}$*; and a point of* $\mathbf{r}(u)$ *is a* circular point *if the circle*

[3]This term is chosen in allusion to the class of a curve, which is the number of tangents that pass through a typical point not on the curve [26, p. 115].

of curvature at that point passes through $\mathbf{p}$*. Equivalently, a point* $\mathbf{r}(u_0)$ *is circular if the center of curvature there coincides with the corresponding point* $\mathbf{b}(u_0)$ *of the untrimmed bisector.*

(To be more precise, an inflection of $\mathbf{r}(u)$ is a point where the curvature *changes sign* — so that $\kappa(u) = 0$ and $\kappa'(u) \neq 0$ or, more generally, the lowest-order non-vanishing derivative of $\kappa(u)$ is *odd*. For example, whereas κ vanishes at $u = 0$ on both $\mathbf{r}(u) = \{u, u^3\}$ and $\mathbf{r} = \{u, u^4\}$, it changes sign in the former case but not in the latter. For simplicity we adhere to the definition given above, thereby occasionally including some points where the circle of curvature does not move from one side of the curve to the other.)

We have already encountered "class" points and "circular" points, in a somewhat different guise (Sections 2.2 and 2.3): they are, respectively, points of $\mathbf{r}(u)$ that induce points at infinity and cusps on the untrimmed bisector. Thus, for fixed u, equations (36) and (37) may be regarded as expressing the condition for the point $\mathbf{p} = (\alpha, \beta)$ to lie on the tangent line to the point $\mathbf{r}(u)$ of a polynomial or rational curve. Likewise, for fixed u, equations (58) and (59) are satisfied when $\mathbf{p} = (\alpha, \beta)$ lies on the circle of curvature at the point $\mathbf{r}(u)$ of a polynomial or rational curve (see Remark 3.2 below).

Theorem 3.1 *For a regular polynomial or rational curve* $\mathbf{r}(u)$ *defined on the interval* $u \in I$*, and a point* $\mathbf{p}$ *not on* $\mathbf{r}(u)$*, let* $\mathbf{b}(u)$ *be the untrimmed bisector of* $\mathbf{p}$ *and* $\mathbf{r}(u)$*, and let* $\{i_1, \ldots, i_M\}$ *be the ordered set of parameter values on* I *that correspond to inflections, class points, and circular points of* $\mathbf{r}(u)$*. Then, denoting the end points of* I *by* i_0 *and* i_{M+1}*, we have either*

$$\mathbf{b}(u) \text{ is active} \quad \text{for all } u \in (i_k, i_{k+1}) \tag{66}$$

or

$$\mathbf{b}(u) \text{ is inactive} \quad \text{for all } u \in (i_k, i_{k+1}) \tag{67}$$

on each span (i_k, i_{k+1}) *for* $k = 0, \ldots, M$*.*

Proof : Consider a point $\mathbf{q}$ moving smoothly along the untrimmed bisector (*i.e.*, such that its parameter value changes smoothly), and let $\mathbf{q}'$ be the corresponding point on the given curve. By definition, in order for $\mathbf{q}$ to change from active to inactive, or vice versa, one of the following must occur: $\mathbf{p}$ must move to a different side of the tangent at $\mathbf{q}'$; the circle of curvature at $\mathbf{q}'$ must move to a different side of the tangent at $\mathbf{q}'$; or $\mathbf{p}$ must move to a different side of the circle of curvature at $\mathbf{q}'$ (*i.e.*, from inside to outside or vice versa). The points of the given curve associated with these occurrences are, respectively, class points, inflections, and circular points. Thus, if $\mathbf{q}$ does not traverse a point whose parameter value corresponds to one of these special points, its status (active/inactive) will remain unchanged. ∎

Remark 3.2 Consider the computation of the endpoints of active segments. Using the formula (45) for the curvature $\kappa(u)$ of $\mathbf{r}(u)$, we see that inflections of the polynomial curve (3) occur at the roots of the polynomial $X'Y''-X''Y'$, while for the rational curve (4) they are roots of $U_1V_2-U_2V_1$ where $(U_1, V_1) = (WX' - W'X, WY' - W'Y)$ and $(U_2, V_2) = (WX'' - W''X, WY'' - W''Y)$. The class points are points of $\mathbf{r}(u)$ where the condition $(\mathbf{p} - \mathbf{r}(u)) \cdot \mathbf{n}(u) = 0$ is satisfied, *i.e.*, $\mathbf{p}$ lies on the tangent line at $\mathbf{r}(u)$ — they are roots of (36) and (37) for polynomial and rational curves, respectively. Finally, recall that circular points on $\mathbf{r}(u)$ are points whose centers of curvature coincide with their corresponding points $\mathbf{b}(u) = \mathbf{r}(u)+d(u)\mathbf{n}(u)$ on the untrimmed bisector. Now at the point $\mathbf{r}(u)$ the circle of curvature has radius $\kappa^{-1}(u)$ and center $\mathbf{e}(u) = \mathbf{r}(u) - \kappa^{-1}(u)\mathbf{n}(u)$, since $\kappa(u)$ is positive when $\mathbf{n}(u)$ points *away* from the center of curvature according to (45); the locus $\mathbf{e}(u)$ is called the *evolute* of $\mathbf{r}(u)$. Thus, the circular points satisfy $\kappa(u) = -1/d(u)$: for polynomial and rational curves, they are roots of (58) and (59).

3.2 Critical points (self–intersections)

In order to make the second — and final — refinement from active segments of the untrimmed bisector to segments of the true bisector, recall that a point $\mathbf{q}$ lies on the true bisector if and only if its associated circle $C_{\mathbf{q}}$ is empty. A boundary between the true bisector and the rest of the untrimmed bisector is marked by a transition from points with empty circles to points with non–empty circles. These transitions are associated with critical points.

Definition 3.4 *A point* $\mathbf{q}$ *of the untrimmed bisector is a* critical point *if the circle* $C_{\mathbf{q}}$ *is tangent to* $\mathbf{r}(u)$ *at two or more points (Figure 11).*

Although defined somewhat differently, critical points are actually self–intersections of the untrimmed bisector.

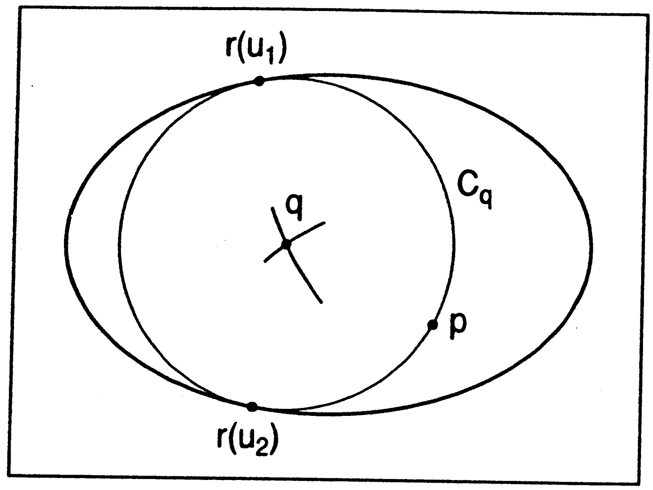

Figure 11: Critical point (self–intersection) of the untrimmed bisector.

Proposition 3.2 *A point* $\mathbf{q}$ *of the untrimmed bisector is a critical point if and only if it is a self–intersection of* $\mathbf{b}(u)$.

Proof : Let $\mathbf{q} \in \mathbf{b}(u)$ be a critical point, and let the circle $C_{\mathbf{q}}$ be tangent to $\mathbf{r}(u)$ at the two points $\mathbf{r}(u_1)$ and $\mathbf{r}(u_2)$. Then the normal at $\mathbf{r}(u_1)$ passes through the center of $C_{\mathbf{q}}$ and, since both $\mathbf{r}(u_1)$ and $\mathbf{p}$ lie on the boundary of this circle, a point of the normal is equidistant from these two points at the center (Figure 11). Thus, $\mathbf{b}(u_1) = \mathbf{q}$. By the same argument, we also have $\mathbf{b}(u_2) = \mathbf{q}$. Hence $\mathbf{q}$ must be a self–intersection of $\mathbf{b}(u)$.

Conversely, let $\mathbf{q} = \mathbf{b}(u_1) = \mathbf{b}(u_2)$ be a self–intersection. Then $\mathbf{r}(u_1)$, $\mathbf{r}(u_2)$, and $\mathbf{p}$ are all equidistant from $\mathbf{q}$. Thus, $\mathbf{r}(u_1)$ and $\mathbf{r}(u_2)$ lie on the circle $C_{\mathbf{q}}$. Moreover, since the curve normals at $\mathbf{r}(u_1)$ and $\mathbf{r}(u_2)$ pass through the center $\mathbf{q}$ of this circle, the circle is also tangent to the curve at the points $\mathbf{r}(u_1)$ and $\mathbf{r}(u_2)$. Hence $\mathbf{q}$ is a critical point. ∎

Proposition 3.2 allows us to trim using self–intersections while arguing the validity of this trim using critical points.

Theorem 3.2 *For a regular polynomial or rational curve* $\mathbf{r}(u)$ *defined on the interval* $u \in I$, *and a point* $\mathbf{p}$ *not on* $\mathbf{r}(u)$, *let* $\mathbf{b}(u)$ *be the untrimmed bisector of* $\mathbf{p}$ *and* $\mathbf{r}(u)$, *and let* $\{i_1, \ldots, i_M\}$ *be the ordered set of parameter values on* I

that correspond to endpoints of active segments on $\mathbf{b}(u)$ *or self–intersections of* $\mathbf{b}(u)$, *i.e.*, $\mathbf{b}(i_j) = \mathbf{b}(i_k)$ *for some* $1 \le j \ne k \le M$. *Then for every active segment* $u \in [i_j, i_k]$ *of* $\mathbf{b}(u)$ *(where* $i_j < i_k$*), we have either*

$$\mathbf{b}(u) \text{ is on the bisector} \quad \text{for all } u \in (i_l, i_{l+1}) \tag{68}$$

or

$$\mathbf{b}(u) \text{ is not on the bisector} \quad \text{for all } u \in (i_l, i_{l+1}) \tag{69}$$

on each span (i_l, i_{l+1}) *for* $l = j, \ldots, k-1$ *between successive self–intersections on that active segment.*

Proof : Let $\mathbf{q}_1 = \mathbf{b}(u_1)$ and $\mathbf{q}_2 = \mathbf{b}(u_2)$ (where $u_1 < u_2$) be two points on an active segment of the untrimmed bisector, neither of them self–intersections, such that $\mathbf{q}_1$ belongs to the bisector but $\mathbf{q}_2$ does not. We will show that the segment $u \in (u_1, u_2)$ of $\mathbf{b}(u)$ must contain a self–intersection (critical point). In other words, if an active segment of $\mathbf{b}(u)$ contains no self–intersections, that entire segment is or is not part of the bisector.

Since $\mathbf{q}_1$ lies on the bisector, $C_{\mathbf{q}_1}$ touches the curve at the point $\mathbf{q}_1' = \mathbf{r}(u_1)$ corresponding to $\mathbf{q}_1$ *and* $C_{\mathbf{q}_1}$ is empty (Remark 2.5). As a point $\mathbf{q}$ moves along the untrimmed bisector from $\mathbf{q}_1$ towards $\mathbf{q}_2$, the radius of the circle $C_{\mathbf{q}}$ changes smoothly while it continues to pass through $\mathbf{p}$, and at all times the curve in the neighborhood of the corresponding point $\mathbf{q}'$ lies completely outside $C_{\mathbf{q}}$ (since $\mathbf{q}$ is active; see Remark 3.1). Now in order for $\mathbf{q}$ to leave the bisector, the circle $C_{\mathbf{q}}$ must become occupied, *i.e.*, the curve must enter $C_{\mathbf{q}}$ (Remark 2.5). We also know that this happens eventually, because $\mathbf{q}_2$ is *not* on the bisector. Since in the neighborhood of $\mathbf{q}'$ the curve does not enter $C_{\mathbf{q}}$, it must first enter $C_{\mathbf{q}}$ at some other point $\mathbf{q}'' \ne \mathbf{q}'$. In order for the curve to enter $C_{\mathbf{q}}$, it must first become tangent to $C_{\mathbf{q}}$, and the location of $\mathbf{q}$ when this occurs is a critical point of the untrimmed bisector, since its circle $C_{\mathbf{q}}$ has *two* points of tangency with the curve — one at $\mathbf{q}'$ and one at the other point $\mathbf{q}''$ where the curve is about to enter the circle (Figure 11). Thus, the segment $u \in (u_1, u_2)$ of $\mathbf{b}(u)$ contains a critical point. ∎

The computation of parameter values that correspond to critical points is rather involved; we defer a full discussion to Section 3.3 below.

Remark 3.3 If the desired interpretation for the bisector of a point $\mathbf{p}$ and a curve segment $S = \mathbf{r}(I)$, defined on a *finite* parameter domain I, is

$$\{\, \mathbf{q} \mid \text{dist}(\mathbf{q},\mathbf{p}) = \text{dist}(\mathbf{q},\mathbf{r}(u)) \text{ and the closest point of } \mathbf{r}(u) \text{ to } \mathbf{q} \text{ lies on } S \,\}\,, \tag{70}$$

rather than its usual meaning

$$\{\, \mathbf{q} \mid \text{dist}(\mathbf{q},\mathbf{p}) = \text{dist}(\mathbf{q},S) \,\}\,, \tag{71}$$

then the statement of Theorem 3.2 is unchanged except that, rather than using just the self-intersections of the segment $\mathbf{b}(I)$, *i.e.*, self-intersections $\mathbf{b}(u_1) = \mathbf{b}(u_2)$ where $u_1, u_2 \in I$, one should use *all* self-intersections of $\mathbf{b}(u)$ that lie on the segment $\mathbf{b}(I)$, *i.e.*, self-intersections $\mathbf{b}(u_1) = \mathbf{b}(u_2)$ where $u_1 \in I$ but u_2 need not lie on I. (The bisector according to definition (70) is generally a *subset* of the "usual" bisector given by (71), since in the former case we wish to prevent $\mathbf{b}(u)$ from encroaching upon "phantom" parts of $\mathbf{r}(u)$ — those that lie *outside* the nominal parameter domain I).

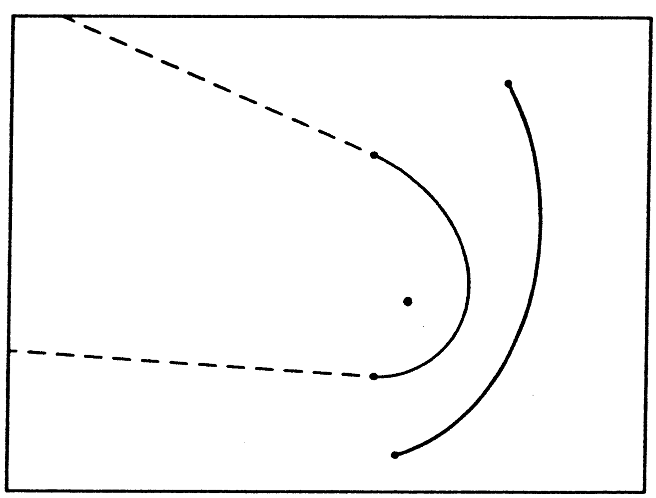

Figure 12: Completing the bisector of a point and a *finite* curve segment.

Note also that when dealing with a finite curve segment $\mathbf{r}(I)$, it will be necessary to "complete" the untrimmed bisector $\mathbf{b}(I)$. This is accomplished by extending along the tangent lines to $\mathbf{b}(I)$ at its affine end points, *i.e.*, end points whose parameter values are *not* roots of (36) or (37), as appropriate; see Figure 12. Recall that the unit tangent at each point of $\mathbf{b}(u)$ is given by expressions (54) and (55). Alternately, one may note that the tangent lines at the endpoints $\mathbf{b}_0$ and $\mathbf{b}_1$ of $\mathbf{b}(I)$ are simply the *perpendicular bisectors* of the given point $\mathbf{p}$ and the corresponding end points, $\mathbf{r}_0$ and $\mathbf{r}_1$, of the segment $\mathbf{r}(I)$. If these tangent–extensions cross each other, it will be necessary to include their point of intersection among the "critical points" in the trimming procedure. The true bisector after trimming may contain all, part, or none of the tangent–extensions to a finite point/curve bisector $\mathbf{b}(I)$.

We now have an algorithm for computing the bisector of a fixed point $\mathbf{p}$ and a regular parametric curve $\mathbf{r}(u)$:

1. Compute the untrimmed bisector of $\mathbf{p}$ and $\mathbf{r}(u)$, as defined by (31) and (35). For polynomial and rational curves, the untrimmed bisector is given by equations (38) and (39), respectively.

2. Find the inflections of the curve $\mathbf{r}(u)$, and its class points (using (36) or (37)) and circular points (using (58) or (59)) with respect to $\mathbf{p}$.

3. For each segment on the untrimmed bisector delineated by these special points, compare the distances of the segment midpoint to $\mathbf{p}$ and $\mathbf{r}(u)$, using (13), and discard the segment if these distances are unequal.

4. Find the critical points (self–intersections) of the untrimmed bisector, using the methods described in Section 3.3 below.

5. Split each remaining segment of the untrimmed bisector at these self–intersections. For each of the resulting segments, compare the distances of the midpoint to $\mathbf{p}$ and $\mathbf{r}(u)$, using (13), and discard the segment if they are unequal. The remaining segments constitute the true bisector.

We conclude this section with some remarks on the general nature of the loci generated by the above algorithm:

Remark 3.4 The true bisector of a point and any (finite or infinite) curve always encloses a convex, simply–connected region of the plane, which may be finite or infinite (see, for example, Figures 13 and 14 below). This is because the region in question can be regarded as the set–intersection of an infinite family of half-planes $\mathcal{S}(u)$ where, for each u, $\mathcal{S}(u)$ contains the given point $\mathbf{p}$ and is bounded by the perpendicular bisector of $\mathbf{p}$ and the curve point $\mathbf{r}(u)$. (The set-intersection of a family of half-planes always yields a convex set; see [16], Theorem 7, p. 112).

3.3 Computing the self-intersections

A self-intersection of the untrimmed bisector $\mathbf{b}(u)$ arises if two (or more) *distinct* parameter values correspond to the same geometric point on its locus (with this definition, we include the exceptional cases where $\mathbf{b}(u)$ just "touches" itself among the self-intersections). Thus, we are interested in identifying all parameter values u that satisfy the vector equation

$$\mathbf{b}(u+\xi) = \mathbf{b}(u) \quad \text{for some } \xi \neq 0\,, \tag{72}$$

where $\mathbf{b}(u) = \{\, X_b(u)/W_b(u), Y_b(u)/W_b(u)\,\}$ is the rational curve defined by equations (38) or (39), as appropriate.

A number of approaches to constructing the minimal polynomial whose roots correspond to self-intersection parameter values are possible. That described below was found empirically to be the most "economical," in the sense of generating the least extraneous intermediate data, and at the same time offers a clear geometric picture. For brevity we discuss only the case of polynomial curves, the rational case being a straightforward but rather tedious generalization of this.

If $\mathbf{b}(u)$ is the untrimmed bisector of the point $\mathbf{p} = (\alpha, \beta)$ and the regular polynomial curve $\mathbf{r}(u) = \{X(u), Y(u)\}$, we know that corresponding points of $\mathbf{b}(u)$ and $\mathbf{r}(u)$ lie on the normal lines to the latter. Given distinct parameter values u and v, we may express the point of intersection (x_i, y_i) of the normal lines at $\mathbf{r}(u)$ and $\mathbf{r}(v)$ in homogeneous coordinates as

$$x_i(u,v) = \frac{X_i(u,v)}{W_i(u,v)} \quad \text{and} \quad y_i(u,v) = \frac{Y_i(u,v)}{W_i(u,v)}\,, \tag{73}$$

where the polynomials X_i, Y_i, and W_i are defined by

$$\begin{aligned}
X_i(u,v) &= Z(u)Y'(v) - Z(v)Y'(u)\,,\\
Y_i(u,v) &= Z(v)X'(u) - Z(u)X'(v)\,,\\
W_i(u,v) &= X'(u)Y'(v) - X'(v)Y'(u)\,.
\end{aligned} \tag{74}$$

For brevity, we have introduced the notation $Z(u) = X(u)X'(u) + Y(u)Y'(u)$ in (74). Note that if $W_i(u,v)$ vanishes, the normal lines at $\mathbf{r}(u)$ and $\mathbf{r}(v)$ are parallel — they intersect in a "point at infinity."

Remark 3.5 Expressions (73–74) fail if $X_i(u,v) = Y_i(u,v) = W_i(u,v) = 0$, a situation that arises when the normal lines at $\mathbf{r}(u)$ and $\mathbf{r}(v)$ are *coincident*. Now the vanishing of *any two* of the polynomials (74) at a point (u_0, v_0) establishes the equality of the ratios

$$X'(u_0) : X'(v_0) \;=\; Y'(u_0) : Y'(v_0) \;=\; Z(u_0) : Z(v_0) \qquad (75)$$

(provided none of these are of the form $0:0$), which in turn indicates that the *third* polynomial must also vanish at (u_0, v_0). Therefore, it suffices to consider any two of (74) if we ensure that none of the ratios (75) are indeterminate. If we set $v = u + \xi$ in (74), discard the trivial solution $\xi = 0$, and then eliminate the variable ξ between the denominator and each component of the numerator, *i.e.*, if we compute the resultants

$$\begin{aligned} \Lambda_x(u) &= \text{Resultant}_\xi\,(\,X_i(u,\xi), W_i(u,\xi)\,)\,, \\ \Lambda_y(u) &= \text{Resultant}_\xi\,(\,Y_i(u,\xi), W_i(u,\xi)\,)\,, \end{aligned} \qquad (76)$$

then the roots of the polynomial

$$\Lambda(u) \;=\; \text{GCD}\,(\,\Lambda_x(u), \Lambda_y(u)\,) \qquad (77)$$

will identify pairs of points on $\mathbf{r}(u)$ that have coincident normal lines. Note that $\Lambda_x(u)$ may have "extraneous" roots, corresponding to the satisfaction of $X_i(u,v) = W_i(u,v) = 0 \neq Y_i(u,v)$ in the degenerate case $Y'(u) = Y'(v) = 0$. Likewise, roots of $\Lambda_y(u)$ may correspond to $Y_i(u,v) = W_i(u,v) = 0 \neq X_i(u,v)$ in the case $X'(u) = X'(v) = 0$. But these extraneous roots of $\Lambda_x(u)$ and $\Lambda_y(u)$ are necessarily distinct when dealing with regular curves, for which it is impossible that $X'(u) = Y'(u) = 0$ or $X'(v) = Y'(v) = 0$. Therefore they cannot be roots of (77). (The vanishing of the "resultant" of two polynomials with respect to any variable expresses a sufficient and necessary condition for them to be satisfied at some value of that variable — see, for example, [25].)

In order for the point (73) to identify a self–intersection of the untrimmed bisector — corresponding to parameter values u and v — (x_i, y_i) must be equidistant from the three points $\mathbf{r}(u)$, $\mathbf{r}(v)$, and $\mathbf{p}$. This condition requires the simultaneous satisfaction of the equations

$$\begin{aligned} [\,x_i - X(u)\,]^2 + [\,y_i - Y(u)\,]^2 &= [\,x_i - \alpha\,]^2 + [\,y_i - \beta\,]^2\,, \\ [\,x_i - X(v)\,]^2 + [\,y_i - Y(v)\,]^2 &= [\,x_i - \alpha\,]^2 + [\,y_i - \beta\,]^2\,. \end{aligned} \tag{78}$$

Cancelling $x_i^2 + y_i^2$ from both sides of these equations and substituting from (73–74), we see that u and v must be simultaneous roots of the polynomials

$$\begin{aligned} \tilde{R}(u,v) &= 2\,[\,\alpha - X(u)\,]\,[\,Z(u)Y'(v) - Z(v)Y'(u)\,] \\ &+ 2\,[\,\beta - Y(u)\,]\,[\,Z(v)X'(u) - Z(u)X'(v)\,] \\ &+ [\,X^2(u) + Y^2(u) - \alpha^2 - \beta^2\,]\,[\,X'(u)Y'(v) - X'(v)Y'(u)\,]\,, \\ \tilde{S}(u,v) &= 2\,[\,\alpha - X(v)\,]\,[\,Z(u)Y'(v) - Z(v)Y'(u)\,] \\ &+ 2\,[\,\beta - Y(v)\,]\,[\,Z(v)X'(u) - Z(u)X'(v)\,] \\ &+ [\,X^2(v) + Y^2(v) - \alpha^2 - \beta^2\,]\,[\,X'(u)Y'(v) - X'(v)Y'(u)\,]\,. \end{aligned} \tag{79}$$

A further reduction of expressions (79) is necessary before proceeding. Each of the three terms comprising $\tilde{R}(u,v)$ and $\tilde{S}(u,v)$ exhibits a factor that vanishes on setting $v = u$. Thus, $v - u$ is an overall factor of both $\tilde{R}(u,v)$ and $\tilde{S}(u,v)$. This factor must be extracted since we are interested only in non–trivial solutions that satisfy $v \neq u$. Hence we take

$$R(u,v) = \frac{\tilde{R}(u,v)}{v-u} \quad \text{and} \quad S(u,v) = \frac{\tilde{S}(u,v)}{v-u} \tag{80}$$

as our working polynomials. We now set $v = u + \xi$ in (80) and re–write these polynomials in the form

$$R(u,\xi) = \sum_{k=0}^{r} a_k(u)\,\xi^k \quad \text{and} \quad S(u,\xi) = \sum_{k=0}^{s} b_k(u)\,\xi^k\,, \tag{81}$$

i.e., as polynomials in ξ whose coefficients are polynomials in u. Then if any particular value of u is to be a parameter value at which a self–intersection occurs, the polynomials (81) must have a common root ξ at that value of u.

The expressions for the coefficients $\{a_k(u)\}$ and $\{b_k(u)\}$ in (81) in terms of $X(u)$, $Y(u)$, and their derivatives and the coordinates α, β of $\mathbf{p}$ are rather complicated, so we shall not present them explicitly here — they are readily derived using a computer algebra system.

Proposition 3.3 *Let the polynomials $a_k(u)$ and $b_k(u)$ be given by (81), and let $\Gamma(u)$ be the polynomial defined in terms of them by the determinant*

$$\Gamma = \begin{vmatrix} a_r & \cdot & \cdot & a_2 & a_1 & a_0 & & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ & & a_r & \cdot & \cdot & a_2 & a_1 & 1 & & \\ & & & a_r & \cdot & \cdot & a_2 & 0 & 1 & \\ & & & & a_r & \cdot & \cdot & 0 & 0 & 1 \\ b_s & \cdot & \cdot & b_2 & b_1 & b_0 & & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ & & b_s & \cdot & \cdot & b_2 & b_1 & 1 & & \\ & & & b_s & \cdot & \cdot & b_2 & 0 & 1 & \\ & & & & b_s & \cdot & \cdot & 0 & 0 & 1 \end{vmatrix}, \tag{82}$$

where there are s rows of a_k's followed by r rows of b_k's and it is understood that blank areas are filled with zeros. Then $\Lambda(u)$ defined by (76) and (77) is a factor of $\Gamma(u)$, and the roots of the polynomial

$$P_i(u) = \frac{\Gamma(u)}{\Lambda(u)} \tag{83}$$

that remains on extracting this factor identify those parameter values at which the untrimmed bisector suffers a self-intersection.

Proof : The resultant of the two polynomials (81) with respect to ξ can be expressed as a Sylvester determinant [25], identical to (82), except that in the last three columns we have replaced each occurrence of a_0 by 1, and each occurrence of a_1 or a_2 by 0. The motivation for this substitution is that the three lowest-order coefficients of ξ in (81) are actually identical

$$a_k(u) \equiv b_k(u) \quad (= c_k(u), \text{ say}) \quad \text{for } k = 0, 1, 2, \tag{84}$$

as can be seen by straightforward but laborious calculation from (79), (80), and (81). Because of this, $[\, c_0(u)\,]^3$ is a factor of the resultant of $R(u,\xi)$ and $S(u,\xi)$ with respect to ξ, and expression (82) is what remains on extracting this factor (by performing elementary column operations on the last three columns of the Sylvester determinant).

Now it may be verified that, apart from a multiplicative constant, $c_0(u)$ is actually identical to the polynomial $P_c(u)$, given by (58), that identifies *cusps* on the untrimmed bisector. This factor must be discarded since we have already dealt with cusps, and are concerned here only with *proper* self-intersections (a cusp may be regarded as the limiting case $u_2 \to u_1$ of a self-intersection $\mathbf{b}(u_2) = \mathbf{b}(u_1)$ with $u_2 \neq u_1$).

Now by construction — see equations (76–77) — roots of the polynomial $\Lambda(u)$ identify pairs of parameter values at which *all three* of the quantities (74) vanish. This implies in turn that expressions (79) and (80) vanish at these values, so $\Lambda(u)$ must be a factor of the resultant of the two polynomials (81) with respect to the variable ξ.

However, we know (by Remark 3.5) that the roots of $\Lambda(u)$ identify pairs of points on $\mathbf{r}(u)$ that have *identical normal lines* — ordinarily, such pairs do *not* yield self-intersections of $\mathbf{b}(u)$. Thus, the factor $\Lambda(u)$ must be eliminated from $\Gamma(u)$. Unfortunately, there does not appear to be any simple *a priori* means of achieving this by modification of the Sylvester determinant.

Paired roots of the remaining polynomial (83) then identify distinct points $\mathbf{r}(u)$ and $\mathbf{r}(v)$ whose normal lines are distinct and intersect in a point (x_i, y_i) equidistant from $\mathbf{r}(u)$, $\mathbf{r}(v)$, and $\mathbf{p}$ — *i.e.*, a self-intersection of $\mathbf{b}(u)$. ∎

There are interesting parallels between the above formulation and the construction of the "self-intersection" polynomial for fixed-distance offsets to a polynomial curve; see Section 4 of [7].

Remark 3.6 A word of caution regarding the polynomial $\Lambda(u)$ is in order. We stated above that points $\mathbf{r}(u)$ and $\mathbf{r}(v)$ of the given curve with *identical normal lines* do not "ordinarily" yield self-intersections of the untrimmed bisector. This is evident if $\mathbf{r}(u)$ and $\mathbf{r}(v)$ are distinct, since for any $\mathbf{q}$ it is impossible that $|\mathbf{q} - \mathbf{p}| = |\mathbf{q} - \mathbf{r}(u)| = |\mathbf{q} - \mathbf{r}(v)|$ with collinear points $\mathbf{r}(u)$, $\mathbf{r}(v)$, and $\mathbf{q}$ when $\mathbf{r}(u) \neq \mathbf{r}(v)$. Exceptionally, however, we may have *coincident* points $\mathbf{r}(u) = \mathbf{r}(v)$ (where $u \neq v$) with *identical normal lines* — these correspond to points where the given curve "touches" itself, and they yield tangential self-intersections of $\mathbf{b}(u)$ identified by roots of $\Lambda(u)$. (For the purpose of trimming, however, these exceptional self-intersections may be ignored, since a transition from points on the true bisector to those not on the true bisector must occur at a critical point, whose circle $C_{\mathbf{q}}$ is tangent to the curve $\mathbf{r}(u)$ at two or more *distinct* points, not simply tangent to the curve at a double point — see the proof of Theorem 3.2).

Note that the distinct real roots $u_1, u_2, \ldots$ of the polynomial $P_i(u)$ satisfy pair-wise correspondences, defined by $\mathbf{b}(u_j) = \mathbf{b}(u_k)$ with $j \neq k$. Some of these may occur "at infinity," *i.e.*, we have $W_b(u_j) = W_b(u_k) = 0$ and $(0,0) \neq (X_b(u_j), Y_b(u_j)) \propto (X_b(u_k), Y_b(u_k))$. Exceptionally, *three or more* of the parameter values $u_1, u_2, \ldots$ may yield the same point on $\mathbf{b}(u)$, identifying a self–intersection of higher multiplicity. Determining these correspondences is not essential to the trimming process, however, since the distance from $\mathbf{r}(u)$ of a point on each arc $u \in [\, u_k, u_{k+1} \,]$ of $\mathbf{b}(u)$ will be explicitly tested.

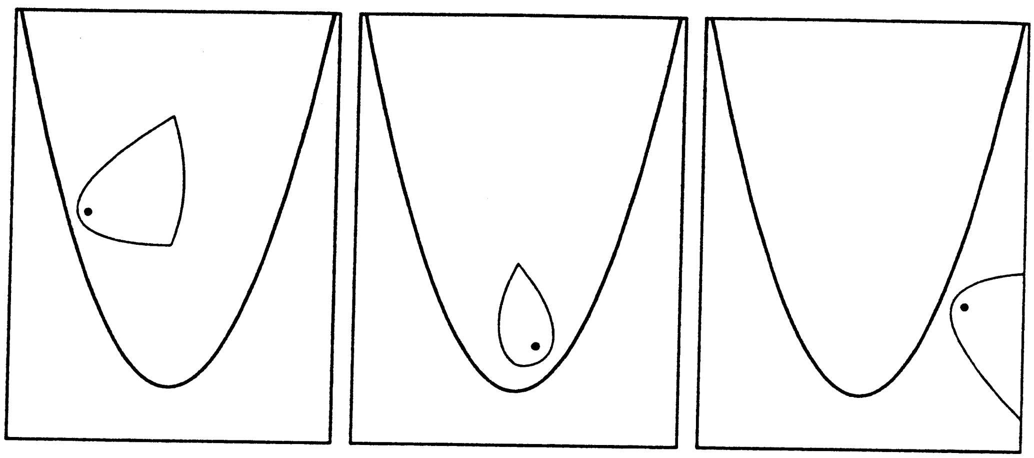

Figure 13: True bisectors of a point and a parabola.

Example 3.1 Consider the "self–intersection" polynomial (83) for the case of Example 2.1 — *i.e.*, the bisector of the point $\mathbf{p} = (\alpha, \beta)$ and the parabola $\mathbf{r}(u) = \{u, u^2\}$. In this case $\Lambda(u)$ is just a constant, while $P_i(u)$ is given by

$$P_i(u) \;=\; u^4 - 2\beta u^2 + \alpha^2 + \beta^2 - \beta\,. \tag{85}$$

Since (85) is of biquadratic form, its roots can be written down explicitly as

$$u \;=\; \pm\sqrt{\beta \pm \sqrt{\beta - \alpha^2}}\,. \tag{86}$$

All four roots are complex if $\alpha^2 > \beta$, *i.e.*, the point (α, β) lies "outside" the parabola. In this case, however, it is necessary to trim the untrimmed bisector at the parameter values (41) corresponding to its points at infinity.

If (α, β) lies "inside" the parabola ($\alpha^2 < \beta$), we have either two or four real roots (and, correspondingly, one or two real self–intersections) according to whether β is less than or greater than $\sqrt{\beta - \alpha^2}$. The latter condition gives

$$\alpha^2 + \beta^2 - \beta = 0 \tag{87}$$

as the locus defining the transition between regimes of one and two self–intersections of the untrimmed bisector as the point (α, β) is varied — this locus is simply the *circle of curvature* at the vertex $(0,0)$ of the parabola. (When (α, β) lies on this circle, $u = 0$ becomes a *double root* of (85) and the corresponding point $(x, y) = (0, \frac{1}{2})$ is an *extraordinary point* of the untrimmed bisector; the remaining roots of (85) are then simply $u = \pm\sqrt{2\beta}$.)

Figure 13 shows the true point/parabola bisectors that correspond to the untrimmed bisectors seen in Figure 6, obtained by trimming the latter at the self–intersection points (86) or points at infinity (41), as appropriate. ∎

Example 3.2 For the ellipse (42), the self–intersection polynomial $P_i(u)$ is the product of two quartic factors given by

$$\begin{aligned} P_{i,1}(u) &= (\alpha^2 + \beta^2 + 2(1-k^2)\alpha + 1 - 2k^2)u^4 \\ &\quad + 2(\alpha^2 + \beta^2 - 1)u^2 + \alpha^2 + \beta^2 - 2(1-k^2)\alpha + 1 - 2k^2 , \\ P_{i,2}(u) &= k(\alpha^2 + \beta^2 - 1)u^4 + 4(1-k^2)\beta u^3 \\ &\quad + 2k(\alpha^2 + \beta^2 - 3 + 2k^2)u^2 + 4(1-k^2)\beta u + k(\alpha^2 + \beta^2 - 1) . \end{aligned} \tag{88}$$

In order to clarify the meaning of these equations, note that (42) has radii of curvature $r = k^2$ at the vertices $u = 0$ and $u = \pm\infty$, and $r = 1/k$ at the vertices $u = +1$ and $u = -1$. Thus, the corresponding circles of curvature are given by the equations

$$\begin{aligned} C_0(x, y) &= (x - 1 + k^2)^2 + y^2 - k^4 = 0 , \\ C_\infty(x, y) &= (x + 1 - k^2)^2 + y^2 - k^4 = 0 , \\ C_{+1}(x, y) &= x^2 + (y^2 - k + 1/k)^2 - 1/k^2 = 0 , \\ C_{-1}(x, y) &= x^2 + (y^2 + k - 1/k)^2 - 1/k^2 = 0 . \end{aligned} \tag{89}$$

Consider first $P_{i,1}(u)$, which we regard as a quadratic in u^2. Making use of the polynomials defined in (89), we see that $P_{i,1}(u)$ can be expressed as

$$P_{i,1}(u) = C_\infty(\alpha, \beta)u^4 + 2(\alpha^2 + \beta^2 - 1)u^2 + C_0(\alpha, \beta) . \tag{90}$$

Evidently there are no real values of u^2 (and hence of u) satisfying $P_{i,1}(u) = 0$ unless its discriminant, given by $\Delta_1 = (\alpha^2 + \beta^2 - 1)^2 - C_\infty(\alpha, \beta)C_0(\alpha, \beta)$, is non–negative. From (89) it can be seen that Δ_1 factors to yield

$$\Delta_1 = 4(1 - k^2)(k^2 - k^2\alpha^2 - \beta^2), \tag{91}$$

and hence the condition $\Delta_1 > 0$ may be interpreted geometrically as follows: either (i) the point (α, β) lies *inside* the ellipse and the x–axis is the major axis of the ellipse ($k < 1$), or (ii) the point (α, β) lies *outside* the ellipse and the y–axis is its major axis ($k > 1$). Exceptionally, we have $\Delta_1 = 0$ when (α, β) lies *on* the ellipse (note that $k \neq 1$ by assumption).

Case (i): $k < 1$ and $k^2\alpha^2 + \beta^2 < k^2$ (major axis along x–axis, point inside ellipse; note that $C_0(x, y)$ and $C_\infty(x, y)$ overlap if $k > 1/\sqrt{2}$, and otherwise are disjoint). Since we are interested only in non–negative roots u^2 to (90), we may invoke Descartes' Law of Signs [25, p. 121] to deduce the number of *positive* roots merely by inspecting its coefficients. Specifically, when $k < 1$ and $k^2\alpha^2 + \beta^2 < k^2$, the middle term of (90) is necessarily negative, and the signature of the coefficients of $P_{i,1}$ is given by:

- $\{+ - +\}$, indicating *two* positive values for u^2, when (α, β) *lies outside both* $C_0(x, y)$ *and* $C_\infty(x, y)$;
- or either $\{- - +\}$ or $\{+ - -\}$, indicating *one* positive value for u^2, when (α, β) *lies inside just one of* $C_0(x, y)$ *and* $C_\infty(x, y)$;
- or (if $k > 1/\sqrt{2}$) $\{- - -\}$, indicating *no* positive values for u^2, when (α, β) *lies inside both* $C_0(x, y)$ *and* $C_\infty(x, y)$.

Exceptionally, there is a real root u^2 to (90) at zero or at infinity if (α, β) lies *on* $C_0(x, y)$ or $C_\infty(x, y)$, respectively.

Case (ii): $k > 1$ and $k^2\alpha^2 + \beta^2 > k^2$ (major axis along y–axis, point outside ellipse — here the circles of curvature $C_0(x, y)$ and $C_\infty(x, y)$ intersect, the ellipse being contained within their common area). In this case the middle term of (90) is necessarily positive, and we again have three possibilities for the number of positive real roots u^2:

- signature $\{- + -\}$, and thus *two* positive roots, if (α, β) *lies inside both* $C_0(x, y)$ *and* $C_\infty(x, y)$ (yet outside the ellipse);

- or signature $\{-++\}$ or $\{++-\}$, and thus *one* positive root, if (α, β) *lies inside just one of* $C_0(x, y)$ *and* $C_\infty(x, y)$;

- or signature $\{+++\}$, and thus *no* positive roots, if (α, β) *lies outside both* $C_0(x, y)$ *and* $C_\infty(x, y)$.

As before, there is a zero or infinite root u^2 to (90) when (α, β) lies *on* $C_0(x, y)$ or $C_\infty(x, y)$, respectively.

(In the above we have glossed over the fact that Descartes' Law actually indicates either *two* or *zero* roots in the case of two sign variations. The latter possibility may be discounted in both Case (i) and Case (ii) by noting that, in instances with two sign variations, the extremum of $P_{i,1}(u^2)$ necessarily occurs at a positive value of u^2 and is opposite in sign to $P_{i,1}(0)$ and $P_{i,1}(\infty)$.)

When $\Delta_1 > 0$, the parameter values that identify self-intersections of the untrimmed bisector are given explicitly by the formula

$$u = \pm\sqrt{\frac{1 - \alpha^2 - \beta^2 \pm \sqrt{\Delta_1}}{C_\infty(\alpha, \beta)}}\,, \qquad (92)$$

which defines zero, one, or two pairs of real values of equal magnitude and opposite sign, according to the criteria enumerated above.

Consider next $P_{i,2}(u)$. We claim that $P_{i,2}(u)$ has real roots (other than zero and infinity) only when $P_{i,1}(u)$ has none, and conversely. To see this, we find by "completing the square" that $P_{i,2}(u)$ may be re-written in the form

$$P_{i,2}(u) = \frac{\left[k(\alpha^2 + \beta^2 - 1)(u^2 + 1) + 2(1 - k^2)\beta u\right]^2 + \Delta_1 u^2}{k(\alpha^2 + \beta^2 - 1)}\,, \qquad (93)$$

where Δ_1 denotes the discriminant (91) introduced previously. For fixed α and β, expression (93) is evidently of constant sign for all real u when $\Delta_1 > 0$, and thus it has no real roots. If $\Delta_1 < 0$, however, the two terms in the numerator of (93) are of opposite sign, and there are real values of u that will cause them to cancel precisely. These values identify the self-intersections of the untrimmed bisector when (i) the point (α, β) lies *outside* the ellipse and the x-axis is the major axis ($k < 1$), or (ii) the point (α, β) lies *inside* and the y-axis is its major axis ($k > 1$).

When $\Delta_1 < 0$, the real roots of $P_{i,2}(u)$ can be computed using the two quadratic equations obtained from (93), namely

$$k(\alpha^2 + \beta^2 - 1)\,(u^2 + 1) + \left[\,2(1 - k^2)\beta \pm \sqrt{-\Delta_1}\,\right]u = 0\,, \qquad (94)$$

whose real solutions are readily determined. However, it will be found — not unexpectedly — that the number and locations of the self-intersections identified by (94), in relation to the ellipse and the untrimmed bisector, are actually identical to those obtained through an appropriate rotation and scaling of the results for $\Delta_1 > 0$. Thus, regardless of the orientation of the ellipse, we may summarize as follows:

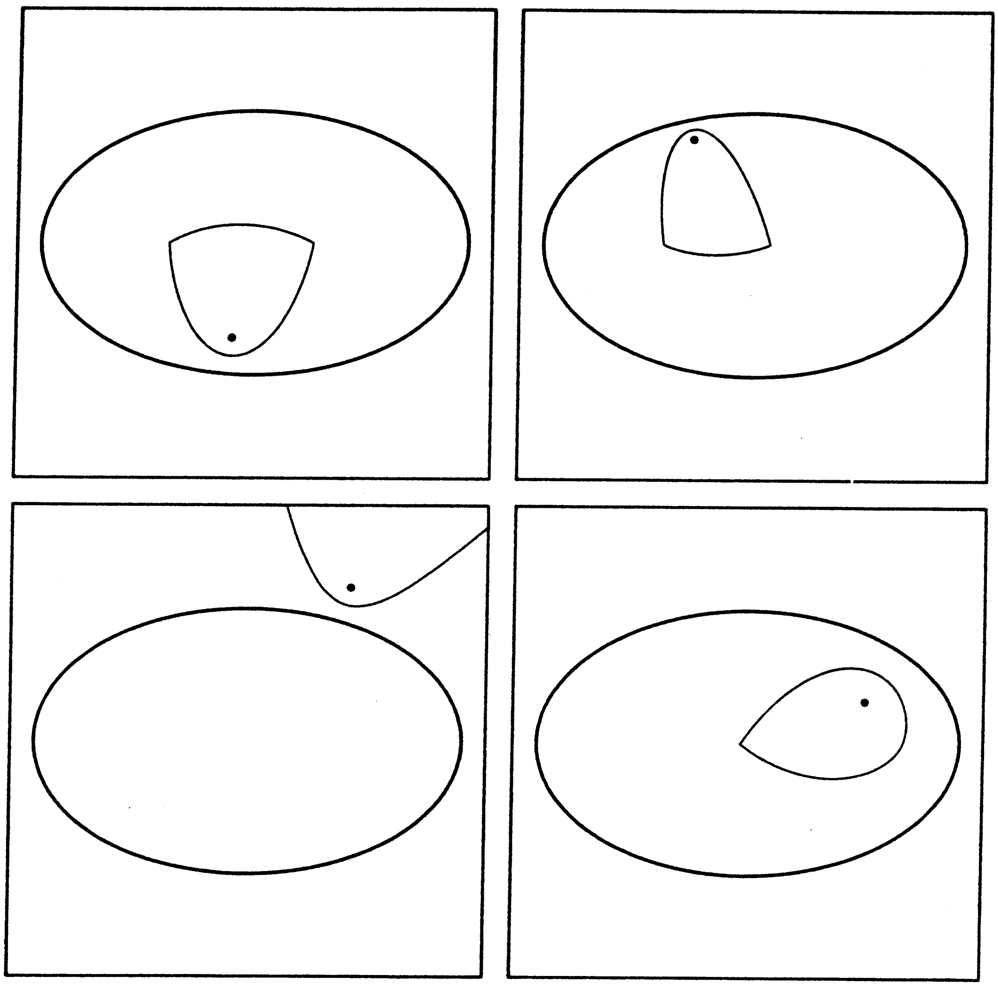

Figure 14: True bisectors of a point and an ellipse.

When (α, β) is located *inside* the ellipse, we will have zero, one, or two self–intersections of the untrimmed bisector, according to whether (α, β) lies inside both, just one, or neither of the circles of curvature to the ellipse at the vertices on its major axis, respectively (the first case being possible only when the circles of curvature actually overlap). If (α, β) is *outside* the ellipse, we have zero, one, or two self–intersections when (α, β) lies inside neither, just one, or both of the circles of curvature to the ellipse at the vertices on its minor axis, respectively. In the latter case, the untrimmed bisector must also be trimmed at its points at infinity, as identified by (44). ∎

The preceding discussion clarifies the behavior seen in Figure 7 above; the trimmed bisectors corresponding to those examples, computed by the methods described above, are shown in Figure 14.

Remark 3.7 Note that for the parabola, $P_i(u)$ is a factor of the polynomial $X_b(u)$ in the parameterization (40) of the untrimmed bisector, while for the ellipse $P_{i,1}(u)$ and $P_{i,2}(u)$ are factors of $Y_b(u)$ and $X_b(u)$ in the parametric form (43). This means that the self–intersections of the untrimmed bisector lie on *axes of symmetry* of the curves under consideration, namely, $x = 0$ in the case of the parabola, and the major or minor axis of the ellipse (according to whether (α, β) lies inside or outside of it).

The above observation is perhaps not unexpected, since we already know that for any self–intersection $\mathbf{q}$ of the untrimmed bisector, the circle $C_{\mathbf{q}}$ (see Definition 2.3) must meet the given curve tangentially in at least two points (Proposition 3.2 and Definition 3.4) — and the center of any inscribed circle of a conic must lie on an axis of symmetry of that conic.

Beyond the realm of conic sections, the degree of the self–intersection polynomial grows very rapidly. For the (polynomial) cubics, for example, a number of examples suggest that $P_i(u)$ is of degree 26 in general, while the extraneous factor $\Lambda(u)$ given by (77) is of degree 6. Also, the polynomials (36) and (58) that define the "class" and "circular" points of $\mathbf{r}(u)$ — *i.e.*, those points that induce points at infinity and cusps on the untrimmed bisector $\mathbf{b}(u)$ — are of degree 4 and 8 in the case of cubics.

Example 3.3 A sobering example of how complicated the self–intersection polynomial can grow is provided by the "simple" cubic $\mathbf{r}(u) = \{u, u^3\}$. In this case the extraneous factor $\Lambda(u)$ is a quartic, while $P_i(u)$ is of degree 24:

$$\begin{aligned}
P_i(u) = {} & 675\,u^{24} + 1620\alpha\,u^{23} - 3888\alpha^2 u^{22} - 8100\beta\,u^{21} \\
& + 540\,(6\alpha\beta - 1)\,u^{20} + 108\alpha\,(144\alpha\beta - 1)\,u^{19} \\
& + 108\,(23\alpha^2 + 235\beta^2)\,u^{18} + 108\,[\,3\alpha(11\alpha^2 - 101\beta^2) - 5\beta\,]\,u^{17} \\
& - 54\,[\,144\alpha^2(\alpha^2 + 3\beta^2) - 100\alpha\beta - 7\,]\,u^{16} \\
& - 108\,[\,\beta(241\alpha^2 + 305\beta^2) + \alpha\,]\,u^{15} \\
& + 108\,[\,36\alpha\beta(5\alpha^2 + 17\beta^2) - 31\alpha^2 + 101\beta^2\,]\,u^{14} \\
& + 36\,[\,432\alpha^2\beta(\alpha^2 + \beta^2) + 12\alpha(7\alpha^2 - 57\beta^2) - 43\beta\,]\,u^{13} \\
& + 2\,[\,27(79\alpha^4 + 894\alpha^2\beta^2 + 303\beta^4) + 5724\alpha\beta - 86\,]\,u^{12} \\
& - 12\,[\,27\alpha(5\alpha^4 + 186\alpha^2\beta^2 + 181\beta^4) + 18\beta(117\alpha^2 + 85\beta^2) - 53\alpha\,]\,u^{11} \\
& - 36\,[\,108\alpha^2(\alpha^2 + \beta^2)^2 - 24\alpha\beta(29\alpha^2 + 45\beta^2) - 7\alpha^2 + 5\beta^2\,]\,u^{10} \\
& - 4\,[\,27\beta(253\alpha^4 + 234\alpha^2\beta^2 - 19\beta^4) + 945\alpha(\alpha^2 + \beta^2) + 293\beta\,]\,u^9 \\
& + 9\,[\,2664\alpha\beta(\alpha^2 + \beta^2)^2 + 12(57\alpha^4 + 234\alpha^2\beta^2 + 65\beta^4) + 520\alpha\beta + 3\,]\,u^8 \\
& - 108\,[\,\alpha(57\alpha^4 + 290\alpha^2\beta^2 + 233\beta^4) + \beta(57\alpha^2 + 41\beta^2) + 2\alpha\,]\,u^7 \\
& + 12\,[\,9(59\alpha^6 + 77\alpha^4\beta^2 - 23\alpha^2\beta^4 - 41\beta^6) \\
& \qquad\qquad + 36\alpha\beta(13\alpha^2 + 17\beta^2) + 63\alpha^2 - 89\beta^2\,]\,u^6 \\
& - 36\,[\,99\alpha(\alpha^2 + \beta^2)^3 + 3\beta(81\alpha^4 + 50\alpha^2\beta^2 - 31\beta^4) \\
& \qquad\qquad + 2\alpha(21\alpha^2 - 19\beta^2)\,]\,u^5 \\
& + 18\,[\,324\alpha\beta(\alpha^2 + \beta^2)^2 + 105\alpha^4 + 210\alpha^2\beta^2 + 41\beta^4\,]\,u^4 \\
& + 4\,[\,243\beta(\alpha^2 + \beta^2)^3 - 54\alpha(7\alpha^4 + 22\alpha^2\beta^2 + 15\beta^4) - 224\beta^3\,]\,u^3 \\
& + 12\,[\,9(7\alpha^6 - 3\alpha^4\beta^2 - 27\alpha^2\beta^4 - 17\beta^6) + 64\alpha\beta^3\,]\,u^2 \\
& - 72\,(\alpha^2 + \beta^2)\,[\,3\alpha(\alpha^2 + \beta^2)^2 - 16\beta^3\,]\,u \\
& + 27(\alpha^2 + \beta^2)^4 - 256\beta^4\,.
\end{aligned} \tag{95}$$

The points at infinity and cusps of the untrimmed bisector are identified by the roots of the polynomials

$$P_\infty(u) = 2u^3 - 3\alpha u^2 + \beta\,,$$

$$P_c(u) = 15u^7 - 27\alpha u^6 + 15\beta u^4 - u^3 + 3\alpha u^2 - 3(\alpha^2 + \beta^2)u + \beta\,. \tag{96}$$

Examples of the untrimmed bisector are shown in Figure 15. ∎

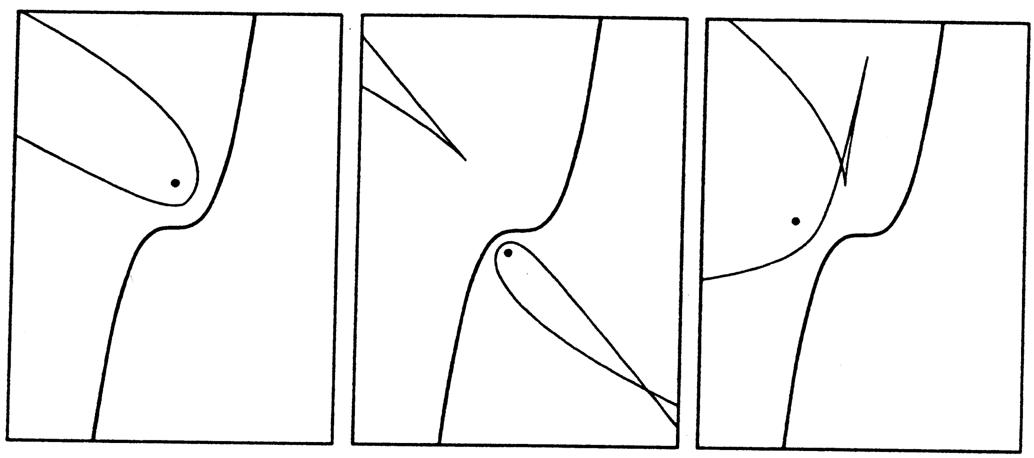

Figure 15: Untrimmed bisectors of a point and $\mathbf{r}(u) = \{u, u^3\}$.

The reader may care to compare equation (95) with the self–intersection polynomial for the constant–distance offset to $\mathbf{r}(u) = \{u, u^3\}$ quoted in [7, equation (100)]. As a further example, we find for the "superbola" $\mathbf{r}(u) = \{u, u^4\}$ that $P_i(u)$ factors into two components, of degree 10 and 48. Figure 16 shows some examples of the untrimmed bisector in this case.

Note that the number of real, affine self–intersections of the untrimmed bisector cannot exceed one–half the degree of the polynomial $P_i(u)$, since by definition two distinct parameter values must be associated with each. In most cases, however, there are far fewer than this, since some roots of $P_i(u)$ correspond to self–intersections of conjugate branches of the complex locus of $\mathbf{b}(u)$, and others to self–intersections "at infinity." Nevertheless, $P_i(u)$ is irreducible in general — there is no simpler representation of just the real, affine self–intersection parameter values.

4 Concluding remarks

The locus of a variable point $\mathbf{q}$, which maintains equal distances with respect to a fixed point $\mathbf{p}$ and a plane polynomial or rational curve $\mathbf{r}(u)$, is amenable to an exact and relatively simple (*i.e.*, rational) parametric representation.

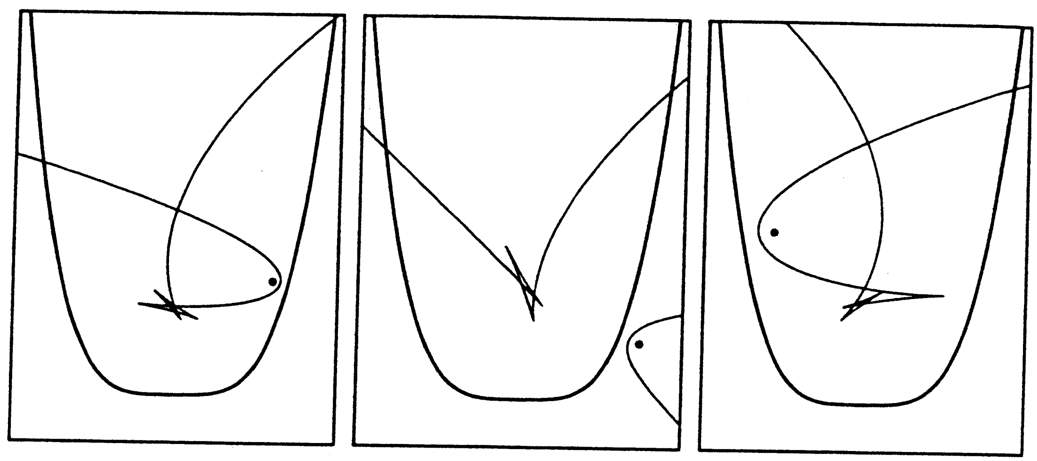

Figure 16: Untrimmed bisectors of a point and $\mathbf{r}(u) = \{u, u^4\}$.

Such point/curve bisectors are thus more compatible with existing computer–aided design systems than other elementary "procedurally–defined" curves — notably the fixed–distance *offsets* to polynomial or rational curves [6, 7], which have no rational parameterizations in general.

The principal difficulty in computing point/curve bisectors undoubtedly lies in the "trimming" procedure, *i.e.*, identifying the parameter values which delimit those subsegments of the untrimmed bisector — see equations (38) and (39) above — that constitute the "true" bisector. Nevertheless, as shown in Section 3, it is possible to attack this problem in an algorithmic manner, and for simple curves (*e.g.*, conics) closed–form analytic expressions for the trim points may even be written down *a priori* (see Examples 3.1 and 3.2).

Once the ordered set of trim points $u_1, u_2, \ldots$ is known, the bisector can be represented by a single rational expression $\mathbf{b}(u)$, restricted to a sequence of disjoint domains $u \in [\, u_1, u_2 \,]$, $u \in [\, u_3, u_4 \,]$, $\ldots$, or by a set of rational Bézier arcs with matched end points — all actually derived from the same parent curve, and mapped individually to the standard domain $u \in [\, 0, 1 \,]$.

The problem of curve/curve bisectors is far more formidable than that of the point/curve bisectors discussed here, even if the given curves have simple parameterizations. We hope to address this problem in future studies.

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